Lecture 14
Analysis of MidTerm Exam
Metrics and Topological Properties I
1. Set theory (6 pt).

Given sets A, B, and C such that \((A \subseteq B) \land (A \cap B \cap C) = \emptyset\), is it always correct that \((A \cap C) = \emptyset\). Draw an illustration.

2. Relations and Mappings (6pt).

Consider a binary relation \(B \subseteq \mathbb{R} \times \mathbb{R}\) such that \(((x, y) \in B) \iff (x \in \mathbb{R}) \land (y \in \mathbb{R}) \land (y = x \cdot |x| / (1 + x^2))\). Is this a mapping \((x \rightarrow y)\)? If yes, define the domain, codomain, range, and find out if it is an injection and a surjection.


Find the truth value of the formula where \(O\) is nonspecific contradiction: \(t(O) = 0\).

4. Algebraic Structures (6pt).

In a set \(S\) of real solutions of the differential equation \(d^4y/dx^4 = -16y(x)\) consider a subset \(Q\) of functions such that \(y(0) = 0\). Is \(Q\) a linear subspace? If yes, find its dimension and basis.

5. Linear Mappings and Matrices (6pt).

Present matrix \(\hat{A} = \begin{pmatrix} 2 & 4 \\ 8 & 6 \end{pmatrix}\) in the diagonal form and find \(\hat{A}^{10}\).
1. Set theory (6 pt).

Given sets $A$, $B$, and $C$ such that $(A \subset B) \land (A \cap B \cap C) = \emptyset$, is it always correct that $(A \cap C) = \emptyset$? Draw an illustration.

Assume $(A \cap C) \neq \emptyset$

$\exists x \ (x \in A) \land (x \in C)$

$(A \subset B) \land (x \in A) \Rightarrow (x \in B)$

$(x \in A) \land (x \in C) \land (x \in B) \Rightarrow (A \cap B \cap C) \neq \emptyset$

Thus the assumption is wrong and $(A \cap C) = \emptyset$.
2. Relations and Mappings (6pt).

Consider a binary relation $B \subseteq \mathbb{R} \times \mathbb{R}$ such that $((x, y) \in B) \iff (x \in \mathbb{R}) \land (y \in \mathbb{R}) \land (y = \frac{x|x|}{1 + x^2})$. Is this a mapping ($x \to y$)? If yes, define the domain, codomain, range, and find out if it is an injection and a surjection.

1. $\forall x \in \mathbb{R}$ $\exists! y \in \mathbb{R}$ $y = \frac{x|x|}{1 + x^2}$ mapping
2. Domain $\mathbb{R}$
3. Codomain $\mathbb{R}$
4. Range?

\[
\lim_{x \to \infty} \frac{x|x|}{1 + x^2} = \lim_{x \to \infty} \frac{x^2}{1 + x^2} = 1
\]

\[
\lim_{x \to -\infty} \frac{x|x|}{1 + x^2} = \lim_{x \to -\infty} \frac{-x^2}{1 + x^2} = -1
\]

$y(x)$ is a continuous function, and $-1 < y(x) < 1$

Range: $(-1, 1)$
5. Injection? 
\[ \frac{dy}{dx} = \begin{cases} 
\frac{2x(1+x^2) - 2x^3}{(1+x^2)^2} & \text{if } x \geq 0 \\
\frac{-2x(1+x^2) + 2x^3}{(1+x^2)^2} & \text{if } x < 0 
\end{cases} \]

The function is monotonous \( \Rightarrow \) injection

6. Not a surjection: Range \( = (-1,1) \neq \mathbb{R} \)
Find the truth value of the formula \( ((A \rightarrow B) \land ((A \land B \land C) \leftrightarrow O)) \rightarrow ((A \land C) \leftrightarrow O) \)
where \( O \) is nonspecific contradiction: \( t(O) = 0 \).

\[
\begin{align*}
t(A \rightarrow B) &= 1 - t(A)(1 - t(B)) = 1 - t(A) + t(A)t(B) \\
t((A \land B \land C) \leftrightarrow O) &= 1 - (t(A)t(b)t(c) - t(O))^2 = 1 - t(A)t(B)t(C) \\
t(D) &= (1 - t(A) + t(A)t(B))(1 - t(A)t(B)t(C)) = \\
&= 1 - t(A)t(B)t(C) - t(A) + t(A)t(B)t(C) + t(A)t(B) - t(A)t(B)t(C) = \\
&= 1 - t(A) + t(A)t(B) - t(A)t(B)t(C) \\
t(E) &= 1 - (t(A)t(C) - t(O))^2 = 1 - t(A)t(C)
\end{align*}
\]
\[ t(Q \rightarrow E) = 1 - (1 - t(A) + t(A) t(B) - t(A) t(B) t(C)) (1 - 1 + t(A) t(C)) = \]
\[ = 1 - t(A) t(C) + t(A) t(C) - t(A) t(B) t(C) + t(A) t(B) t(C) = \]
\[ = 1 \]
4. Algebraic Structures (6pt).

In a set $S$ of real solutions of the differential equation $\frac{d^4y}{dx^4} = -16y(x)$ consider a subset $Q$ of functions such that $y(0) = 0$. Is $Q$ a linear subspace? If yes, find its dimension and basis.

Assume

$$\frac{d^4y_1}{dx^4} = -16y_1(x) \quad y_1(0) = 0 \quad y_1(x) \in Q$$

$$\frac{d^4y_2}{dx^4} = -16y_2(x) \quad y_2(0) = 0 \quad y_2(x) \in Q$$

Then

$$\frac{d^4}{dx^4}\left(c_1y_1(x) + c_2y_2(x)\right) = -16\left(c_1y_1(x) + c_2y_2(x)\right) \quad c_1y_1(0) + c_2y_2(0) = 0$$

Thus $c_1y_1(x) + c_2y_2(x) \in Q$, $Q$ is a linear subspace.
All real solutions of \( \frac{dy}{dx} = -16y(x) \)

Characteristic roots \( \lambda^4 = -16 \), \( \lambda = \sqrt{2}(\pm 1 \pm i) \)

\[ y(x) = C_1 e^{\sqrt{2}x} \sin \sqrt{2}x + C_2 e^{\sqrt{2}x} \cos \sqrt{2}x + C_3 e^{-\sqrt{2}x} \sin \sqrt{2}x + C_4 e^{-\sqrt{2}x} \cos \sqrt{2}x \]

\( y(0) = 0 \): Any \( C_1 \), any \( C_3 \), and \( C_2 + C_4 = 0 \) \( \iff \) \( C_4 = -C_2 \)

\[ y(x) = C_1 e^{\sqrt{2}x} \sin \sqrt{2}x + C_2 \left( e^{\sqrt{2}x} - e^{-\sqrt{2}x} \right) \cos \sqrt{2}x + C_3 e^{-\sqrt{2}x} \sin \sqrt{2}x \]

Subspace is 3-dimensional and the basis can be chosen in a form

\[ \hat{v}_1 = e^{\sqrt{2}x} \sin \sqrt{2}x \quad \hat{v}_2 = \left( e^{\sqrt{2}x} - e^{-\sqrt{2}x} \right) \cos \sqrt{2}x \quad \hat{v}_3 = e^{-\sqrt{2}x} \sin \sqrt{2}x \]
5. Linear Mappings and Matrices (6pt).

Present matrix \( \hat{A} = \begin{pmatrix} 2 & 4 \\ 8 & 6 \end{pmatrix} \) in the diagonal form and find \( \hat{A}^{10} \).

Eigenvalues: \( (2-\lambda)(6-\lambda) - 32 = 0 \) \( \lambda^2 - 8\lambda + 12 - 32 = 0 \) \( \lambda = 4 \pm \sqrt{16 + 20} \) \( \lambda_{1,2} = -2, 10 \)

Eigen vectors: \( \lambda = -2 \) \( \begin{pmatrix} 4 & 4 \\ 8 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) \( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)

\( \lambda = 10 \) \( \begin{pmatrix} -8 & 4 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) \( \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \)

Diagonalization matrix \( S = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \) \( S^{-1} = \frac{1}{2+1} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \)

Diagonal form of \( A \): \( A = S \Lambda S^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \)
$$A^{10} = S \Lambda^{10} S^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} (-2)^{10} \\ 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2^{10} & -2^{10} \\ 10^{10} & 10^{10} \end{pmatrix} =$$

$$= \frac{1}{3} \begin{pmatrix} 2^{10} + 10^{10} & -2^{10} + 10^{10} \\ -2^{10} + 2 \cdot 10^{10} & 2^{10} + 2 \cdot 10^{10} \end{pmatrix}$$
Necessity of Metric and Distance

In a set theory:

\[(a \in A) \land (b \in A) \land (c \in A)\]

Comparisons such as \(a = b\) or \(b \neq c\) are possible.

Can you compare the elements? Can you say \(a > b\)? – Metric norm is needed.

In case of \((a \neq b) \land (b \neq c) \land (c \neq a)\) can you say “a is close to b” or “a is closer to be b than to c”? – Metric distance is needed.

“Real-world” problems

Math optimizations: For a given mapping \(A \rightarrow B\) (that is, \(\forall a \in A \exists! b \in B \ b = f(a)\)), find the element in \(A\) that corresponds to the largest (or smallest) possible \(b = f(a)\).

Fitting: Given the element \(a \in A\), among the elements \(b \in B \subset A\) find the best approximation for the element \(a\).
Distance

Metric distance:

For an arbitrary set $V$, consider a mapping

$$V \times V \rightarrow \mathbb{R} \quad \text{(that is, } \forall (v_1, v_2) \in V \times V \; \exists! \ r \in \mathbb{R} \quad r = \rho(v_1, v_2) \ )$$

with following properties

1) The zero-distance axiom: The metric distance is equal to zero if and only if the elements are identical

$$\left( \rho(v_1, v_2) = 0 \right) \iff (v_1 = v_2)$$

2) The symmetry axiom: The metric distance between the elements $v_1$ and $v_2$ is the same as between $v_2$ and $v_1$.

$$\rho(v_1, v_2) = \rho(v_2, v_1)$$

3) The triangular axiom:

$$\rho(v_1, v_3) \leq \rho(v_1, v_2) + \rho(v_2, v_3)$$
Distance (continued)

The metric distance, if exists, is always a non-negative number.

Proof: in the triangular axiom consider $v_3 = v_1$ and use the symmetry axiom:

\[
\rho(v_1, v_3) \leq \rho(v_1, v_2) + \rho(v_2, v_3)
\]

\[
0 = \rho(v_1, v_1) \leq \rho(v_1, v_2) + \rho(v_2, v_1) = 2 \rho(v_1, v_2)
\]

\[
0 \leq \rho(v_1, v_2)
\]

Very often, the requirement for the distance to be non-negative is included into the set of axioms, but it is not necessary as long as it follows from other axioms.
Metric Space

A set with metric distance is called the metric space.

The metric space does not have to be a linear space (the term “space” is a bit misleading)

Examples of metric spaces.

1) Trivial metric distance: the concept of elements to be equal or to be not equal in the set theory expressed in terms of distance

\[ \rho(v_1, v_2) = \begin{cases} 
  0 & \text{if } v_1 = v_2 \\
  1 & \text{if } v_1 \neq v_2 
\end{cases} \]

Check that all the distance axioms are satisfied.

Note that such distance concept is useless as long as it does not bring anything new to the set theory.
2) $V = \mathbb{N}$ (Note that natural numbers do not form a linear space: $1 - 2 = -1 \notin \mathbb{N}$)

$$\rho(n_1, n_2) = |n_1 - n_2|$$

3) $V = \mathbb{R}$

$$\rho(r_1, r_2) = |r_1 - r_2|$$

4) $V$ is an $n$-dimensional linear space (mapped into $\mathbb{C}^n$)

$$v_1 = \sum_{i=1}^{n} \alpha_i \tilde{v}_i \quad v_2 = \sum_{i=1}^{n} \beta_i \tilde{v}_i \quad \rho(v_1, v_2) = \sqrt[p]{\sum_{i=1}^{n} |\alpha_i - \beta_i|^p}$$

Notice that (a) there is a freedom in choosing the order $p$, and (b) the distance defined this way depends on the choice of the basis vectors.

$p=2$ is the case of Euclidian metric distance – fits well the intuitive concept of distance.

It would be convenient to have a basis-independent definition of distance – additional restriction must be imposed on the basis ("normalization procedure") or, otherwise, the metric formula itself must contain the normalizing factors.
Metric Spaces, Continued

5) $V = F = \text{“set of all functions } f(x) \text{ defined on the interval } [0, 1]$$

This is an infinite-dimensional linear space

A possible way to introduce a distance metric:

- Define a grid of points $x_i = i/n$, $0 \leq i \leq n$.
- Find an averaged distance between the corresponding points for the functions $f_1(x)$ and $f_2(x)$:

$$\rho_F(f_1, f_2) = \sqrt{\frac{1}{n} \sum_{i=0}^{n} |f_1(x_i) - f_2(x_i)|^2}$$

- Find limit when $n \to \infty$

$$\rho_F(f_1(x), f_2(x)) = \sqrt{\frac{1}{0} \int_{0}^{1} |f_1(x) - f_2(x)|^2 \, dx}$$

This procedure, generally speaking, requires the functions to be continuous.
Metric Spaces, Continued

Other distance definitions in \( F \):

\[
\rho_{F_1}(f_1(x), f_2(x)) = \int_{0}^{1} |f_1(x) - f_2(x)| \, dx
\]

\[
\rho_{F_2}(f_1(x), f_2(x)) = \max \left( |f_1(x) - f_2(x)| \right) \quad (x \in [0, 1])
\]
6) \( V = P_2 = \) “set of all polynomials of degree two or less defined on the interval \([0, 1]\)”

\[
p_1 = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \quad p_2 = \beta_0 + \beta_1 x + \beta_2 x^2
\]

As long as there is a unique one-to-one mapping from \( V \) to \( \mathbb{C}^3 \), the distance can be defined as distance between the corresponding elements in \( \mathbb{C}^3 \), for example,

\[
\rho(p_1, p_2) = \sqrt{\sum_{i=1}^{3} |\alpha_i - \beta_i|^2}
\]

Although \( P_2 \subseteq F \) (polynomials are also functions, see the previous example), this distance definition is quite different.
Metric Spaces, Continued

Consider $f_1(x) = 0 = 0 + 0 \cdot x + 0 \cdot x^2$

$f_2(x) = x = 0 + 1 \cdot x + 0 \cdot x^2$

$f_3(x) = x^2 = 0 + 0 \cdot x + 1 \cdot x^2$

F–distance:

$$\rho_{F}(f_1, f_2) = \sqrt{\int_0^1 |0 - x|^2 \, dx} = \sqrt{\frac{x^3}{3}} \bigg|_0^1 = \frac{1}{\sqrt{3}}$$

$$\rho_{F}(f_1, f_3) = \sqrt{\int_0^1 |0 - x^2|^2 \, dx} = \sqrt{\frac{x^5}{5}} \bigg|_0^1 = \frac{1}{\sqrt{5}}$$

$$\rho_{F}(f_1, f_2) > \rho_{F}(f_1, f_3)$$

"Distance between $f_1$ and $f_2$ is larger than distance between $f_1$ and $f_3"$

P–distance

$$\rho_{P_2}(f_1, f_2) = \sqrt{0^2 + 1^2 + 0^2} = 1$$

$$\rho_{P_2}(f_1, f_3) = \sqrt{0^2 + 0^2 + 1^2} = 1$$

$$\rho_{P_2}(f_1, f_2) = \rho_{P_2}(f_1, f_3)$$

"Distance between $f_1$ and $f_2$ is equal to the distance between $f_1$ and $f_3"
Metric Spaces, Continued

Another distance definition in $P_2$:

$$
\rho_{P_2}(p_1, p_2) = \max\left(\alpha_1 - \beta_1, \alpha_2 - \beta_2, \alpha_3 - \beta_3\right)
$$

$$
\rho_{P_2}(f_1, f_2) = \max(0, 1, 0) = 1
$$

$$
\rho_{P_2}(f_1, f_3) = \max(0, 0, 1) = 1
$$

$$
\rho_{P_2}(f_2, f_3) = \max(0, 1, 1) = 1
$$

However, in accordance with the previous definition

$$
\rho_{P_2}(f_1, f_2) = \sqrt{0^2 + 1^2 + 0^2} = 1
$$

$$
\rho_{P_2}(f_1, f_3) = \sqrt{0^2 + 0^2 + 1^2} = 1
$$

$$
\rho_{P_2}(f_2, f_3) = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}
$$
No lecture on Monday, October 20
(on travel to NSF)