Lecture 20
Operator Equations I
Sample Problem on Fourier Series

Problem III. Find the Fourier series of the function determined on the interval \([0, 1]\)

\[ f(x) = \begin{cases} 
  x, & x < \frac{1}{2} \\
  4x^3, & x > \frac{1}{2} 
\end{cases} \]

Solution. \( f(x) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi i n x} \), where \( C_n = \langle f(x), e_n \rangle = \int_0^1 f(x) e^{-2\pi i n x} \, dx = \int_0^{\frac{1}{2}} x e^{-2\pi i n x} \, dx + \int_{\frac{1}{2}}^1 4x^3 e^{-2\pi i n x} \, dx \)

\[ n = 0 \implies C_0 = \int_0^1 f(x) \, dx = \int_0^{\frac{1}{2}} x \, dx + \int_{\frac{1}{2}}^1 4x^3 \, dx = \frac{17}{16} \]

\[ n \neq 0 \implies C_n = \frac{2i}{\pi n} + \frac{11 - (-1)^n}{4} \frac{2}{(\pi n)^2} + \frac{-3i + 3(-1)^n}{2} \frac{2}{(\pi n)^3} + \frac{-3 + 3(-1)^n}{2} \frac{2}{(\pi n)^4} \]
Check it

$$n := -1000 \ldots 1000$$

$$C_n := \begin{cases} \frac{17}{16} & \text{if } n = 0 \\ \frac{2i}{\pi \cdot n} + \frac{11}{4} - \frac{(-1)^n}{2} + \frac{-3i + \frac{3i}{2} \cdot (-1)^n}{(\pi \cdot n)^2} + \frac{-\frac{3}{2} + \frac{3}{2} \cdot (-1)^n}{(\pi \cdot n)^3} + \frac{-\frac{3}{2} + \frac{3}{2} \cdot (-1)^n}{(\pi \cdot n)^4} & \text{otherwise} \end{cases}$$

$$f(x) := \sum_{n} \left( C_n \cdot e^{2i \cdot \pi \cdot n \cdot x} \right)$$
Operator Equations: General Notes

In elementary mathematics, equations such as \( f(x) = y \) are considered. Here \( x \) and \( y \) are numbers, and \( f(x) \) is a function (a mapping \( \mathbb{R} \rightarrow \mathbb{R} \)).

Solving this equation is related to finding the inverse function \( x = f^{-1}(y) \), which does not necessarily exists, or, due to multiple possible solutions cannot be defined unambiguously.

In some cases exact solutions are difficult to find, but numerical algorithm can be developed (such as Newton’s method) to find an approximate solution with any pre-assigned accuracy.

Similar concepts are applicable to equations \( f(v) = w \) that represent a mapping \( F \) from a set \( V \) to a set \( W \) (\( V \rightarrow W \)). The sets’ elements are not necessarily numbers: they could be functions, geometrical figures, symmetry transforms, etc.

The concept of approximate solutions will definitely require evaluation of how large the difference between the set elements is, that is, the distance (the sets to be metric spaces).
Inverse Mappings and Operator Equations

Let $F$ be a mapping $V \rightarrow W$ from a set $V$ to a set $W$: $\forall v \in V \exists! w \in W \ w = f(v)$.

$f(v)$ is often called the operator performing the mapping $F$.

The equation $w = f(v)$ relating an unknown $v \in V$ to a given $w \in W$ is called an operator equation.

A formal solution, if it always exists, and if it is always unique, is given by an inverse mapping $v = f^{-1}(w)$

By considering the nature of the sets $V$ and $W$ and the mapping $F$, one should be able to judge:

1) If the solution exists in the first place;

2) Whether this solution is unique;

3) How to find the exact solution. If this is not possible, how to find the approximate solution (in case of metric spaces).
Typical Problems

Consider a mapping of a Banach space into itself.

Recall: a Banach space is a closed linear space with norm defined. It’s always a metric space. For a linear space, zero element $\theta$ is defined, as well as operations such as multiplying an element by a complex number and adding the elements.

Typical problems:

1) Find the root(s) $f(v) = \theta$ e.g. $d^2y/dx^2 = 0$

2) Find the fixed point(s) $f(v) = v$ e.g. $d^2y/dx^2 = y(x)$

3) Find the eigenvalue(s) and eigenvector(s) $f(v) = \lambda v$ e.g. $d^2y/dx^2 = \lambda y(x)$

In essence, these problems are identical

E. g. finding fix points vs. finding roots:

$(f(v) = v) \iff (f(v) - v = \theta) \iff (g(v) = \theta)$, where $g(v) = f(v) - v$

$(f(v) = \theta) \iff (f(v) + v = v) \iff (g(v) = v)$, where $g(v) = f(v) + v$
The Eigenvectors Problem

The eigenvectors / eigenvalues problem is a bit different, because there are simultaneously two related unknowns: $v \in V$ and $\lambda \in \mathbb{C}$. However, in a cross-product set $VC = V \times \mathbb{C}$, the eigenvectors / eigenvalues problem can be reduced to a root-finding problem.

The norm in $VC$ can be defined as follows:

$$\forall (v, c) \in VC \; (v \in V \land c \in \mathbb{C}) \; ||(v, c)|| = (||v||^2 + ||c||^2)^{1/2}$$

Then, the eigenvector / eigenvalues problem can be re-formulated

Consider a mapping $S \; VC \rightarrow VC$ defined as $s((v, c)) = (f(v) - c \cdot v, ||v|| - 1)$

Note that $f(v) - c \cdot v \in V$ and $||v|| - 1 \in \mathbb{R} \subset \mathbb{C}$, so that $S$ is indeed a mapping from $VC$ to $VC$.

$s((v, c)) = \theta_{VC} \; (\theta_{VC} = (\theta_v, 0 + 0i))$ is equivalent to

$f(v) - c \cdot v = \theta_v \; \text{and} \; ||v|| = 1$

The second condition $||v|| = 1$ guarantees that $f(v) - c \cdot v = \theta_v$ is satisfied for a non-zero $v$. 
Contraction Mapping Theorem
(Banach Fixed-Point Therem)

The fixed-point problem is convenient for analysis as long as there exist a criterion for the operator equation to have a unique solution.

Let V be a complete metric space, and F be a contraction mapping of V into itself. Then F has one and only fixed point (\( f(v) = v \) has one and only one solution)

Proof:

Consider the sequence of consecutive mappings under F (the Picard sequence):

Take any \( v_0 \in V \), then \( v_1 = f(v_0), v_2 = f(v_1), v_3 = f(v_2), \ldots, v_n = f(v_{n-1}), \ldots \)

The elements in the sequence are also called the Picard iterations.

Regardless of the starting element \( v_0 \), the sequence is converging, and the limit is a unique solution of the operator equation \( f(v) = v \).
Convergence Proof

The contraction mapping: \( \exists \mu \ 0<\mu<1 \quad \rho(f(x), f(y)) \leq \mu \rho(x, y) \)

For a given starting element \( v_0 \) find the distance to the corresponding element \( f(v_0) \)
\( \rho(v_0, f(v_0)) = d. \) For the Picard sequence

\[
\begin{align*}
  f(v_0) &= v_1 & \rho(v_0, v_1) &= \rho(v_0, f(v_0)) = d \\
  f(v_1) &= v_2 & \rho(v_1, v_2) &= \rho(f(v_0), f(v_1)) \leq \mu \cdot \rho(v_0, v_1) = \mu d \\
  f(v_2) &= v_3 & \rho(v_2, v_3) &= \rho(f(v_1), f(v_2)) \leq \mu \cdot \rho(v_1, v_2) \leq \mu^2 d \\
  f(v_n) &= v_{n+1} & \rho(v_n, v_{n+1}) &= \rho(f(v_{n-1}), f(v_n)) \leq \mu \cdot \rho(v_{n-1}, v_{n-2}) \leq \ldots \leq \mu^n d
\end{align*}
\]

Let us show that the Picard sequence is the Cauchy sequence.

Take any small \( \varepsilon > 0. \) Try to find such \( N(\varepsilon) \) so that for any \( m, n > N \) the distance
\( \rho(v_n, v_m) < \varepsilon. \) To be certain, let \( m > n. \)

\[
\begin{align*}
  \rho(v_n, v_m) &\leq \rho(v_m, v_{m-1}) + \rho(v_{m-1}, v_n) \leq \rho(v_m, v_{m-1}) + \rho(v_{m-1}, v_{m-2}) + \rho(v_{m-2}, v_n) \leq \\
  &\leq \rho(v_m, v_{m-1}) + \rho(v_{m-1}, v_{m-2}) + \rho(v_{m-2}, v_{m-3}) + \ldots + \rho(v_{n+1}, v_n) \leq \\
  &\leq \mu^{m-1} d + \mu^{m-2} d + \ldots + \mu^n d = \mu^n d (1 + \mu + \mu^2 + \ldots + \mu^{m-n-1}) \leq \mu^n d / (1 - \mu)
\end{align*}
\]

Thus, an appropriate \( N(\varepsilon) \) would be
\[
N(\varepsilon) = \left\lfloor \frac{\ln(\varepsilon(1-\mu)/d)}{\ln \mu} \right\rfloor
\]
Uniqueness

Being a Cauchy sequence, the Pickard sequence must always converge to a limiting element. The space V is closed, thus the limit belongs to V.

The limiting element is the fixed point of the mapping F. Once the contraction mapping is a continuous mapping

$$\tilde{v} = \lim_{n \to \infty} v_n \quad f(\tilde{v}) = f\left(\lim_{n \to \infty} v_n\right) = \lim_{n \to \infty} f(v_n) = \lim_{n \to \infty} v_{n+1} = \tilde{v}$$

In order to prove uniqueness, assume that there are two fixed points under the mapping F.

$$v^# = f(v^#) \quad \text{and} \quad v^\$ = f(v^\$) \quad (v^# \neq v^\$$)

Consider the distance $\rho(v^#, v^\$) = \rho(f(v^#), f(v^\$)) \leq \mu \rho(v^#, v^\$)$

$$\rho(v^#, v^\$)(1 - \mu) \leq 0$$

However, from distance definition $\rho(v^#, v^\$) \geq 0$, thus $\rho(v^#, v^\$)(1 - \mu) \geq 0$.

The only possibility to satisfy both inequalities is $\rho(v^#, v^\$) = 0$, which implies $v^# - v^\$ = \theta$
Successive Approximations

The Picard sequence is a tool to find approximate solutions. The error of a successive approximations is estimated at

$$\rho(v_n, \tilde{v}) \leq \mu \rho(v_n, v_{n-1})(1 + \mu + \mu^2 + ...) = \mu \rho(v_n, v_{n-1}) \frac{1}{1 - \mu}$$

If $\mu << 1$, the convergence is fast.

If $\mu \approx 1$ (still must be $\mu < 1$ to have a contraction mapping), the convergence is slow, that is, many iterations would be required to achieve a desirable accuracy.

If $\mu > 1$, the Picard sequence may be diverging. Only occasionally, when the starting point is close enough to the solution of the fixed-point problem and locally, in a vicinity of the solutions, conditions for the contraction mapping are satisfied ($\mu_{\text{local}} < 1$), the convergence is possible.
Example of a Picard Sequence

\[ f(x) = \cos(x) \quad x \in [0,1] \]

In general, for functions that have a continuous derivative

\[ \forall x_1, x_2 \quad \exists x_3 \in [x_1, x_2] \quad \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \left. \frac{df}{dx} \right|_{x=x_3} \]

Thus \( \mu = \max_{x \in [0,1]} \left| \frac{df}{dx} \right| = \max_{x \in [0,1]} |\sin(x)| = \sin(1) < 1 \)

The fixed-point problem: \( f(x) = x \quad \cos(x) = x \)

The Picard sequence \( x_{n+1} = \cos(x_n) \)

\( x_0 = 0 \quad x_1 = \cos(0) = 1 \quad x_2 = \cos(1) \quad x_3 = \cos(\cos(1)) \quad x_4 = \cos(\cos(\cos(1))) \ldots \)
Picard Sequence for Non-contracting Mapping

\[ f(x) = e^{-2x} \quad x \in [0,1] \]
\[ \mu = \max_{x \in [0,1]} \left| \frac{df}{dx} \right| = \max_{x \in [0,1]} \left| -2e^{-2x} \right| = 2 > 1 \]

The Picard sequence, nevertheless converges
\[ x_0 = 0 \quad x_1 = 1 \quad x_2 \approx 0.135 \quad x_3 = 0.763 \quad \tilde{x} \approx 0.42630275 \]

This happens because locally, in the vicinity of the fixed point \[ \left| \frac{df}{dx} \right| \approx \left| -2e^{\tilde{x}} \right| = 2\tilde{x} \approx 0.8526055 < 1 \]

In other words, even though the mapping is not contracting in the entire range \( x \in [0, 1] \), it is contracting in a vicinity of the fixed point.

For such cases, however, in general, the very existence of the solution and its uniqueness is not guaranteed.

A similar example: \( f(x) = e^{-4x} \) even though the solution exists \( x \approx 0.30054197 \), the Picard sequence is diverging.
\[ x_0 = 0.3 \quad x_1 = 0.3012 \quad x_2 = 0.2998 \quad x_3 = 0.3015 \ldots \]
Picard sequence in case of non-contracting mapping

\[ f(x) = \exp(-2x) \]

\[ f(x) = \exp(-4x) \]
Application to Differential Equations

The Picard sequence: does it always converge? No

is this the only way to solve the equation? No

is this the best (the fastest) way to solve the equation? No

then why?

The Picard sequence in the case of a contracting mapping establishes sufficient condition for existence and uniqueness of the solution. This is a useful tool for theoretical studies: namely, to prove the existence and uniqueness of the solution.

Then, the solution can be found using other approaches. Sometimes, just by guessing and checking, or by writing a “reasonably looking formula” and then adjusting the free parameters in it. Finally, based on the proved uniqueness, the solution that has bee found is the unique solution.

Example: prove the uniqueness of the solution of a differential equation.

Once again, the Picard sequence is not a practical way to solve the differential equations. Instead, this is a tool to proof the existence and uniqueness of the solution.
Basic form of an ordinary differential equations of n-th order: 
\[ f\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \ldots, \frac{d^ny}{dt^n}\right) = 0 \]

In some cases the equation can be resolved with respect to the highest derivative:
\[ \frac{d^n y}{dt^n} = g\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}\right) \]

This equation can be presented as a first-order vector differential equations.

Consider a vector \( \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y(t) \\ dy/dt \\ \vdots \\ d^{n-1}y/dt^{n-1} \end{pmatrix} \)

The n-th order differential equation is equivalent to

\[ \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = g\left(t, \begin{pmatrix} y_1 \\ \frac{dy}{dt} \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}} \end{pmatrix}\right) \quad \text{or just} \quad \frac{d\vec{y}}{dt} = g(t, \vec{y}(t)) \]
Differential Equations, Continued

The simplest case – one-dimensional vectors $\bar{y}$.
Let us reduce the problem to the fixed-point problem.

To be specific, $\frac{dy}{dt} = g(t, y)$ $\quad t \in [0, 1] \quad y(0) = y_0$

$y(t) \in V$, a Banach space of functions that have a derivative on $[0, 1]$.

$$\frac{dy}{dt} = g(t, y)$$

$$\int_0^t \frac{dy}{dt} dt = \int_0^t g(t', y(t')) dt'$$

$$y(t) - y(0) = \int_0^t g(t', y(t')) dt'$$

$$y(t) = \hat{h}(y(t)) = y_0 + \int_0^t g(t', y(t')) dt'$$

The last equation is the fixed-point equation.
The Contraction Criterion

\[ \rho(\hat{h}(y_1(t)), \hat{h}(y_2(t))) = \| \hat{h}(y_1(t)) - \hat{h}(y_2(t)) \| = \int_0^1 |\dot{h}(y_1(t)) - \dot{h}(y_2(t))| dt = \]

\[= \int_0^1 \int_0^t g(t', y_1(t')) - g(t', y_2(t')) dt' dt \leq \int_0^1 \left( \int_0^t g(t', y_1(t')) - g(t', y_2(t')) dt' \right) dt \]

estimation for: \[|g(t', y_1(t')) - g(t', y_2(t'))| \leq \max_{t \in [0,1]} \left| \frac{dg}{dy} \right| |y_1(t) - y_2(t)| \]

Then, integrate over \( t' \in [0,1] \):

\[ \| \hat{h}(y_1(t)) - \hat{h}(y_2(t)) \| \leq \max_{t \in [0,1]} \left| \frac{dg}{dy} \right| \| y_1(t) - y_2(t) \| \int_0^1 dt \]

If it appears that \( \max_{t \in [0,1]} \left| \frac{dg}{dy} \right| < 1 \), the mapping is contracting, and every Picard sequence will converge to the unique solution.

If it appears that \( \max_{t \in [0,1]} \left| \frac{dg}{dy} \right| > 1 \), the iterative solution is not possible over entire interval \([0,1]\)

But, taking sufficiently small \( \delta > 0 \) such that \( \max_{t \in [0,1]} \left| \frac{dg}{dy} \right| \int_0^\delta dt = \delta \cdot \max_{t \in [0,1]} \left| \frac{dg}{dy} \right| \), one can find a unique solution over the interval \([0, \delta]\). Then take \( y(\delta) \) as initial condition for the next interval \( t \in [\delta, 2\delta] \) e.t.c.

If \( \max_{t \in [0,1]} \left| \frac{dg}{dy} \right| < \infty \), suitable value of \( \delta \) is, e.g. \( \frac{1}{2} \left( \frac{1}{\max_{t \in [0,1]} \left| \frac{dg}{dy} \right|} \right) \).
Generalization for the case $n > 1$

\[
\frac{d^n y}{dt^n} = g\left(t, y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \ldots, \frac{d^{n-1} y}{dt^{n-1}}\right)
\]

\[
\tilde{y} = \begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{pmatrix}
= \begin{pmatrix}
y(t) \\
\frac{dy}{dt} \\
\vdots \\
\frac{d^{n-1} y}{dt^{n-1}}
\end{pmatrix}
\]

\[
\frac{d\tilde{y}}{dt} = g(t, \tilde{y}(t))
\]

Initial condition to be specified: \( \tilde{y}(0) = \begin{pmatrix}
y_1(0) \\
y_2(0) \\
\vdots \\
y_n(0)
\end{pmatrix} = \begin{pmatrix}
y(0) \\
\frac{dy}{dt} \big|_{t=0} \\
\vdots \\
\frac{d^{n-1} y}{dt^{n-1}} \big|_{t=0}
\end{pmatrix} \)
Note on Singularity

Consider equation \[ \frac{dy}{dt} = g(t, y), \quad y(t_0) = y_0 \]

If \( g \) and \( \frac{\partial g}{\partial y} \) are continuous in a neighborhood of the point \( (t_0, y_0) \), then there exist a continuous solution \( y(t) \) of the differential equation \( y' = g(t, y) \) satisfying the initial condition \( y(t_0) = y_0 \) and this solution is unique.

This is a consequence of the Picard’s theorem that establishes existence and uniqueness of a solution for a fixed-point problem \( v = F(v) \) defined by a contraction mapping \( F \) that maps a complete normed space into itself. Every Picard sequence – a sequence of elements obtained by consecutive application of the mapping starting from an arbitrary initial element (that is, \( v_0, v_1 = F(v_0), v_2 = F(v_1), \ldots \) ) – appears to be converging to a unique limit regardless the choice of the initial element.

When function \( g(t, y) \) or \( \frac{\partial g}{\partial y} \) is singular at a given point \( (t_0, y_0) \), that is

\[
\lim_{t \to t_0} g(t, y) = \infty \quad \text{or} \quad \lim_{t \to t_0} \frac{\partial g}{\partial y} = \infty
\]

the differential equation reformulated as a fixed-point problem is linked to a mapping which is not a contraction, and thus the Picard theorem is not applied – the solution does not necessarily exist and, if it does exist, it is not necessarily unique. Pay attention to the points of singularity.