Lecture 16
Banach and Hilbert Spaces – I
Banach and Hilbert Spaces I

General Introduction

A set with metric **distance** – metric space.

A **linear space** with metric distance – normed space

A **complete linear space** with metric distance - Banach space

Linearity allows for evaluation of the set elements (a mapping with domain V) and a “natural” definition of distance (a mapping with domain \(V \times V\))

Due to linearity, following becomes possible

\[
\text{norm } ||v|| = \text{distance } (v, \emptyset)
\]

\[
\text{distance } (v_1, v_2) = \text{norm } ||v_1 - v_2||
\]
Norm Mapping

Norm is a mapping $V \rightarrow \mathbb{R}$ (V is a linear space) $\forall v \in V \exists! ||v|| \in \mathbb{R}$ such that

1) $||v|| \geq 0$

2) $(||v|| = 0) \iff (v = \theta)$

3) $\forall v \in V \forall \alpha \in \mathbb{C} \ ||\alpha \cdot v|| = |\alpha| \cdot ||v||$

4) $||v_1 + v_2|| \leq ||v_1|| + ||v_2||$

A linear space with norm mapping is called a normed space.

The norm definition is very similar to the distance definition.

Notice, however: Norm mapping $V \rightarrow \mathbb{R}$

Distance mapping $V \times V \rightarrow \mathbb{R}$

In a sense, the norm definition is “simpler” – the domain is V rather that $V \times V$. 
Banach Space

Once norm is defined, the distance can be defined as follows

$$\rho(v_1, v_2) = ||v_1 - v_2||$$

It is called norm-induced metric distance (or, “naturally defined” distance).

Other metric distance definitions are possible, but it is convenient to work with the norm-induced metric distance.

Note:

Not every distance is norm-induced.

A possible definition of distance in $\mathbb{R}$: $\rho(r_1, r_2) = |\text{atan}(r_2) - \text{atan}(r_1)|$

Practical case for this norm:

angular distance

However, $|\text{atan}(r) - \text{atan}(0)| = |\text{atan}(r)|$ is not a norm

Complete normed space is called Banach space.

Completeness is essential as it allows for definition of continuous mapping.
Examples of Banach Spaces

1. Norm in \( \mathbb{R}^n \) (Euclidian space)

\[
\vec{v} = \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\vdots \\
\alpha_n
\end{pmatrix}
\]

\[
\|\vec{v}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \ldots + \alpha_n^2}
\]

2. The linear space of continuous functions on [0, 1] is not a Banach space because it is not complete (a sequence of continuous functions may converge to a limit which is not a continuous function).

3. The linear space of all functions on [0, 1] is a Banach space.
Norm: a measure of magnitude

The concept of norm allows to compare “apples to oranges”

Once the norm is defined in different sets, the set elements can be compared one to another, even though they might be of very different nature.

What is bigger? – whatever has a larger norm.

\[
\vec{v} = \begin{pmatrix} 1+i \\ 1-i \\ 1 \end{pmatrix} \in \mathbb{C}^3
\]

\[
f(x) = \sin(\pi x) \quad x \in [0, 1]
\]

\[
\|\vec{v}\| = \sqrt{|1+i|^2 + |1-i|^2 + 1^2} = \sqrt{2+2+1} = \sqrt{5}
\]

\[
\|f\| = \sqrt{\int_0^1 \sin^2(\pi x) \, dx} = \sqrt{\frac{1}{2}}
\]

\[
\|\vec{v}\| > \|f\|
\]
Subspaces in Banach Space

Let V be a Banach space (complete linear space with norm).

Let \((v_1, v_2, v_3, \ldots, v_m, \ldots)\) be a Cauchy sequence in V with the limit

\[ \bar{v} = \lim_{m \to \infty} v_m \in V \quad \text{(due to completeness)} \]

\[ \forall \varepsilon > 0 \quad \exists M(\varepsilon) \quad \forall m > M \quad \rho(v_m, \bar{v}) = \|v_m - \bar{v}\| < \varepsilon \]

It implies for the norm-induced metrics

\[ \forall \varepsilon > 0 \quad \exists M(\varepsilon) \quad \forall m > M \quad \rho(v_m - \bar{v}, \theta) = \|v_m - \bar{v}\| < \varepsilon \]

Thus \((v_1 - \bar{v}, v_2 - \bar{v}, v_3 - \bar{v}, \ldots, v_m - \bar{v}, \ldots)\) is also a Cauchy sequence. It converges to \(\theta\).

Given the basis \(\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \ldots, \tilde{v}_n\}\) in V,

\[ v_m = \sum_{i=1}^{n} \alpha_i^{(m)} \tilde{v}_i \]

\[ \bar{v} = \sum_{i=1}^{n} \bar{\alpha}_i \tilde{v}_i \]

\[ v_m - \bar{v} = \sum_{i=1}^{n} \left( \alpha_i^{(m)} - \bar{\alpha}_i \right) \tilde{v}_i \]
Subspaces, continued

\[
\lim_{m \to \infty} (v_m - \tilde{v}) = \lim_{m \to \infty} \sum_{i=1}^{n} (\alpha_i^{(m)} - \bar{\alpha}_i) \tilde{v}_i = \sum_{i=1}^{n} \lim_{m \to \infty} (\alpha_i^{(m)} - \bar{\alpha}_i) \tilde{v}_i = \theta
\]

\(\tilde{v}_i\) are linearly independent, thus

\[
\lim_{m \to \infty} (\alpha_i^{(m)} - \bar{\alpha}_i) = 0 \quad \text{and} \quad \lim_{m \to \infty} \alpha_i^{(m)} = \bar{\alpha}_i
\]

For a finite-dimensional Banach space:

If a sequence of elements is converging to some limit element, then all the sequences of coefficients are converging numerical sequences. Moreover, they converge to corresponding coefficients of the decomposition of the limit element over the chosen basis.

Corollary: a finite-dimensional subspace of a Banach space is complete, thus it is a Banach space.
Subspaces, continued

Similar consideration is possible for a space with infinite (but countable) basis

\[ v = \sum_{i=1}^{\infty} \alpha_i \tilde{v}_i = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \alpha_i \tilde{v}_i \right) = \text{limit of partial sums} \]

\( \tilde{v}_i \ (i = 1, 2, 3, \ldots \infty) \) form a basis in \( V \) if

1) All of them are linearly independent

2) Any \( v \in V \) can be presented as an (infinite) linear combination of the basis elements

A procedure to find the coefficients of the decomposition \( \alpha_i \) is still to be defined, both for the case of a finite-dimensional space as well as for the infinite-dimensional space.
Inner Product

Metric distance – “a measure of how different the elements of a set are”

Norm – “a measure of how large the element is” (The set must be a linear space)

Inner product - “a measure of to what degree the elements are linearly independent”

Inner product (dot product, scalar product): a mapping $V \times V \rightarrow \mathbb{C}$ such that

1) Not-negative value $\langle v, v \rangle \geq 0$
2) Zero if and only if $(\langle v, v \rangle = 0) \Leftrightarrow (v = \theta)$
3) Asymmetry $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle^*$
4) Distributivity $\langle v_1, v_2 + v_3 \rangle = \langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle$
5) Scaling $\langle \alpha \cdot v_1, v_2 \rangle = \alpha \cdot \langle v_1, v_2 \rangle$ $(\forall \alpha \in \mathbb{C})$

By comparison with the norm axioms, a norm can be defined as a particular case of an inner product

$$\|v\| = \sqrt{\langle v, v \rangle}$$
Note on norms, distances, and inner products

The norm, metric distance, and inner product can, in principle, be defined independently. However, it is convenient to take one primary definition (the inner product \( \langle v_1, v_2 \rangle \)), construct corresponding norm \( \| v \| = \sqrt{\langle v, v \rangle} \), and then distance \( \rho(v_1, v_2) = \| v_1 - v_2 \| \).

This will be a unified approach to judge "how large elements of a set are," "how different they are," and "how linearly independent they are."

Hierarchy of metrics: inner product \( \rightarrow \) norm \( \rightarrow \) distance

However,
If it is known that a given distance formula is generated by a norm mapping, the formula for the norm can be restored: \( \| v \| = \rho(v, \theta) \).

If it is known that a given norm is generated by an inner product, the formula for the inner product can be restored.
Given the norm mapping, which is known to be induced by an inner product, the inner product can be expressed using the formulas for the norm

\[
\|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \langle y, x \rangle = \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \langle x, y \rangle^* = \|x\|^2 + \|y\|^2 + 2 \text{Re} \langle x, y \rangle
\]

\[
\|x-y\|^2 = \langle x-y, x-y \rangle = \|x\|^2 + \|y\|^2 - \langle x, y \rangle - \langle y, x \rangle = \|x\|^2 + \|y\|^2 - \langle x, y \rangle - \langle x, y \rangle^* = \|x\|^2 + \|y\|^2 - 2 \text{Re} \langle x, y \rangle
\]

\[
\|x+y\|^2 - \|x-y\|^2 = 4 \text{Re} \langle x, y \rangle
\]

\[
\text{Re} \langle x, y \rangle = \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2)
\]
\[ \|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 \|y\|^2 + \langle x, y \rangle + \langle y, x \rangle = \|x\|^2 + \|y\|^2 + \text{Re}(\langle x, y \rangle) \]

\[ \|x - y\|^2 = \langle x - y, x - y \rangle = \|x\|^2 \|y\|^2 - \langle x, y \rangle + \langle y, x \rangle = \|x\|^2 + \|y\|^2 - \text{Re}(\langle x, y \rangle) \]

\[ \|x - y\|^2 - \|x + y\|^2 = 4 \text{Im}(\langle x, y \rangle) \]

\[ \text{Im}(\langle x, y \rangle) = \frac{1}{4} (\|x - y\|^2 - \|x + y\|^2) \]

\[ \langle x, y \rangle = \text{Re}(\langle x, y \rangle) + i \text{Im}(\langle x, y \rangle) = \]

\[ = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) + \frac{i}{4} (\|x - y\|^2 - \|x + y\|^2) \]
Important to remember:

Not every norm mapping is induced by an inner product, and not every distance is induced by a norm mapping.

Example of a norm mapping that is not induced by an inner product:
$$||a + ib|| = \max(|a|, |b|)$$

Example of a distance mapping that is not induced by a norm:
$$\rho(r_1, r_2) = |\arctan(r_2) - \arctan(r_1)|$$
Hilbert Space

A complete linear space with inner product is called a Hilbert space.

Example: A real 3-dimensional Euclidean space with inner product

\[
\langle \vec{v}_1, \vec{v}_2 \rangle = |\vec{v}_1| \cdot |\vec{v}_2| \cdot \cos \varphi(\vec{v}_1, \vec{v}_2)
\]

In a Cartesian coordinate system

\[
\langle \vec{v}_1, \vec{v}_2 \rangle = v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z}
\]

A more general case: vectors in \( \mathbb{C}^n \)

\[
\langle \vec{\alpha}, \vec{\beta} \rangle = \alpha_1 \beta_1^* + \alpha_2 \beta_2^* + \alpha_3 \beta_3^* + ... + \alpha_n \beta_n^*
\]

Example of an infinite-dimensional Hilbert space: complex-valued functions on \([0, 1]\)

with inner product \( \langle f_1(x), f_2(x) \rangle = \int_0^1 f_1(x)f_2^*(x)dx \)

Strictly speaking, it requires

\[
\langle f_1(x), f_2(x) \rangle < \infty \quad \forall f_1(x), f_2(x)
\]

in particular, \( \int_0^1 |f(x)|^2 dx < \infty \)
Example of a norm and distance generated by the inner product

In the Euclidian space, the inner product is $\langle \vec{v}_1, \vec{v}_2 \rangle = v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z}$

Then, the norm becomes $\| \vec{v} \| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{v_xv_x + v_yv_y + v_zv_z} = \sqrt{v_x^2 + v_y^2 + v_z^2}$

And the distance $\rho(\vec{v}_1, \vec{v}_2) = \| \vec{v}_1 - \vec{v}_2 \| = \sqrt{(v_{1x} - v_{2x})^2 + (v_{1y} - v_{2y})^2 + (v_{1z} - v_{2z})^2}$
Another example

Let the inner product be \( \langle \vec{v}_1, \vec{v}_2 \rangle = v_{1x}v_{2x} + v_{1y}v_{2y} + \frac{1}{2}v_{1x}v_{2y} + \frac{1}{2}v_{1y}v_{2x} \)

Then, the norm becomes \( \|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{v_xv_x + v_yv_y + \frac{1}{2}v_xv_y + \frac{1}{2}v_yv_x} = \sqrt{v_x^2 + v_y^2 + v_xv_y} \)

And the distance \( \rho(\vec{v}_1, \vec{v}_2) = \|\vec{v}_1 - \vec{v}_2\| = \sqrt{(v_{1x} - v_{2x})^2 + (v_{1y} - v_{2y})^2 + (v_{1x} - v_{2x})(v_{1y} - v_{2y})} \)

Geometric interpretation
The Cauchy-Schwartz Inequality

For any inner product
\[ |\langle v_1, v_2 \rangle| \leq \sqrt{\langle v_1, v_1 \rangle} \cdot \sqrt{\langle v_2, v_2 \rangle} \]

This is what allows for introducing a norm
(and then, a norm-induced metric distance)
based on the inner product.
Using the norm notations
\[ |\langle v_1, v_2 \rangle| \leq \|v_1\| \cdot \|v_2\| \]

Note, from the scaling axiom
\[ \langle \alpha \cdot v, v \rangle = \alpha \cdot \langle v, v \rangle = \alpha \cdot \sqrt{\langle v, v \rangle} \cdot \sqrt{\langle v, v \rangle} = \sqrt{\langle \alpha \cdot v, \alpha \cdot v \rangle} \cdot \sqrt{\langle v, v \rangle} = \|\alpha \cdot v\| \cdot \|v\| \]
and this this the only case for the inequality in the C-Sch inequality.
Cauchy-Schwartz Inequality, Cont-d

Proof for the Cauchy-Schwartz inequality.

General scheme: For given \( v_1 \) and \( v_2 \), consider a linear combination

\[
\ vec{v}_3 = \ vec{v}_1 + c \ vec{v}_2.
\]

\( \forall c \langle \ vec{v}_3, \ vec{v}_3 \rangle \geq 0 \), moreover, as a rule \( \langle \ vec{v}_3, \ vec{v}_3 \rangle > 0 \), unless \( \ vec{v}_3 = \theta \) (then \( \langle \ vec{v}_3, \ vec{v}_3 \rangle = 0 \))

Choose appropriate value of a constant \( c \) to prove the Cauchy-Schwartz inequality.

\[
\langle \ vec{v}_3, \ vec{v}_3 \rangle = \langle \ vec{v}_1 + c \ vec{v}_2, \ vec{v}_1 + c \ vec{v}_2 \rangle \geq 0
\]

\[
\langle \ vec{v}_1, \ vec{v}_1 \rangle + c \langle \ vec{v}_2, \ vec{v}_1 \rangle + c^* \langle \ vec{v}_1, \ vec{v}_2 \rangle + |c|^2 \langle \ vec{v}_2, \ vec{v}_2 \rangle \geq 0
\]

\[
\langle \ vec{v}_1, \ vec{v}_1 \rangle + c \langle \ vec{v}_1, \ vec{v}_2 \rangle^* + c^* \langle \ vec{v}_1, \ vec{v}_2 \rangle + |c|^2 \langle \ vec{v}_2, \ vec{v}_2 \rangle \geq 0 \quad \text{take } c = -\frac{\langle \ vec{v}_1, \ vec{v}_2 \rangle}{\langle \ vec{v}_2, \ vec{v}_2 \rangle}
\]

\[
\langle \ vec{v}_1, \ vec{v}_1 \rangle - \frac{\langle \ vec{v}_1, \ vec{v}_2 \rangle}{\langle \ vec{v}_2, \ vec{v}_2 \rangle} \langle \ vec{v}_1, \ vec{v}_2 \rangle^* - \frac{\langle \ vec{v}_1, \ vec{v}_2 \rangle^*}{\langle \ vec{v}_2, \ vec{v}_2 \rangle} \langle \ vec{v}_1, \ vec{v}_2 \rangle + |c|^2 \langle \ vec{v}_2, \ vec{v}_2 \rangle \geq 0
\]

\[
\langle \ vec{v}_1, \ vec{v}_1 \rangle - \frac{|\langle \ vec{v}_1, \ vec{v}_2 \rangle|^2}{\langle \ vec{v}_2, \ vec{v}_2 \rangle} \geq 0 \quad \langle \ vec{v}_1, \ vec{v}_1 \rangle \langle \ vec{v}_2, \ vec{v}_2 \rangle \geq |\langle \ vec{v}_1, \ vec{v}_2 \rangle|^2
\]
Cauchy-Schwartz Inequality, Examples

\[ \left| \int_{a}^{b} f_1(x) f_2^*(x) \, dx \right| \leq \sqrt{\int_{a}^{b} |f_1(x)|^2 \, dx} \cdot \sqrt{\int_{a}^{b} |f_2(x)|^2 \, dx} \]

\[ \left| \sum_{i=1}^{n} \alpha_i \beta_i^* \right| \leq \sqrt{\sum_{i=1}^{n} |\alpha_i|^2} \cdot \sqrt{\sum_{i=1}^{n} |\beta_i|^2} \]
Sample problem: Norm

With the norm defined as \( \|f\| = \sqrt[p]{\int_{-1}^{1}|f(x)|^p \, dx} \), \( p \in \mathbb{N} \), which of the elements has a larger norm: 
\( f_1(x) = 2|x|, x \in [-1, 1] \) or \( f_2(x) = 2 - 2|x|, x \in [-1, 1] \) ? Does the answer depend on the particular value of \( p \)?

\[
\|f_1\| = \sqrt[p]{\int_{-1}^{1}(2|x|)^p \, dx} = \sqrt[p]{\int_{0}^{1}(2 \cdot 2^p \cdot x^p) \, dx} = 2 \sqrt[p]{\left. \frac{x^{p+1}}{p+1} \right|_0^1} = 2 \sqrt[p]{\frac{2}{p+1}}
\]

\[
\|f_2\| = \sqrt[p]{\int_{-1}^{1}(2 - 2|x|)^p \, dx} = \sqrt[p]{\int_{0}^{1}(2 - 2x)^p \, dx} = 2 \sqrt[p]{\left. \frac{(1 - x)^{p+1}}{p+1} \right|_0^1} = 2 \sqrt[p]{\frac{2}{p+1}}
\]

\[
1 - x = t \\
x = 0 \rightarrow t = 1 \\
x = 1 \rightarrow t = 0
\]

\(-dx = dt\)

\[
\|f_2\| = 2 \sqrt[p]{\int_{1}^{0} t^p \, dt} = 2 \sqrt[p]{\left. \frac{t^{p+1}}{p+1} \right|_0^1} = 2 \sqrt[p]{\frac{2}{p+1}}
\]

\[
\|f_1\| = \|f_2\| \quad \forall p \in \mathbb{N}
\]
Sample problem: Inner Product

In $\mathbb{R}^3$, a norm mapping is proposed in the form $n(x,y,z) = \max(|x|, |y|, |z|)$. Is there an inner product that would generate this norm? Hint: in a real space, if such an inner product exists, it must be $\langle a, b \rangle = \left( \frac{\|a + b\|^2 - \|a - b\|^2}{4} \right)$. 

Check if $\langle a, b \rangle = \left( \frac{\|a + b\|^2 - \|a - b\|^2}{4} \right)$ is a formula for inner product.

1) $\langle a, a \rangle = \left( \frac{\|a + a\|^2 - \|a - a\|^2}{4} \right) = \frac{\|2a\|^2}{4} = \|a\|^2 \geq 0 \quad \text{OK}$

2) $\langle a, a \rangle = 0 \iff \|a\|^2 = 0 \iff a = \theta \quad \text{OK}$

3) $\langle a, b \rangle = \left( \frac{\|a + b\|^2 - \|a - b\|^2}{4} \right)$
   $\langle b, a \rangle = \left( \frac{\|b + a\|^2 - \|b - a\|^2}{4} \right)$
   $\langle a, b \rangle = \langle b, a \rangle = \langle b, a \rangle^*$ \quad \text{OK}
4) \( \langle a+b, c \rangle = \langle a, c \rangle + \langle b, c \rangle \)

\[
\langle a+b, c \rangle = \frac{(\| a+b+c \|^2 - \| a+b-c \|^2)}{4}
\]

\[
\langle a, c \rangle + \langle b, c \rangle = \frac{(\| a+c \|^2 - \| a-c \|^2 + \| b+c \|^2 - \| b-c \|^2)}{4}
\]

Not necessarily satisfied.

E.g. take \( a = b = (1, 1, 0) \), \( c = (1, 0, 0) \):

\[
\frac{(\| a+b+c \|^2 - \| a+b-c \|^2)}{4} = \frac{(\| (3, 2, 0) \|^2 - \| (1, 2, 0) \|^2)}{4} = \frac{(3^2 - 2^2)}{4} = \frac{5}{4}
\]

\[
\frac{(\| a+c \|^2 - \| a-c \|^2 + \| b+c \|^2 - \| b-c \|^2)}{4} = 2 \cdot \frac{(\| a+c \|^2 - \| a-c \|^2)}{4} = 2 \cdot \frac{(\| (2, 1, 0) \|^2 - \| (0, 1, 0) \|^2)}{4} = \frac{2}{4} \left( 2^2 - 1^2 \right) = \frac{3}{2} \neq \frac{5}{4}
\]

Thus, there can be no inner product that would generate

the norm \( \| (x, y, z) \| = \max (|x|, |y|, |z|) \)