Lecture 2
Basic Set Theory
Set Theory, Relations and Mappings, Math Logic: Why here?

To define the legitimate language of math theories.

These are very general theories built axiomatically.

Any rigorous math theory uses them, often implicitly or without direct reference. E. g. “Natural numbers form a subset of real numbers” (What is a set? What is a subset?) “Cartesian coordinates on a plane are represented by an ordered pair of real numbers” (What is an ordered pair? Is it a set?)

What does it mean to prove a mathematical statement (a theorem)? – To reduce it to the level of axioms and definitions. There must a convention of what is excepted to be truth and what kind of logical steps in derivations are considered to be legitimate.

Every topic in mathematics consists in defining new objects, classifying them (defining sets), and defining operations with the objects (set relations and mappings) – this is why rigorous set theory is extremely important for mathematics.
Examples

Since \( a = b \), thus \( a^2 = b^2 \) \hspace{1cm} \text{true}

Since \( a^2 = b^2 \), thus \( a = b \) \hspace{1cm} \text{false}

Since \( a^2 = b^2 \) and \( a>0 \) and \( b>0 \), thus \( a = b \) \hspace{1cm} \text{true}

Since \( a^2 = b^2 \), thus \( a = b \) or \( a = -b \) \hspace{1cm} \text{true}

Since \( a^4 = b^4 \), thus \( a = b \) or \( a = -b \) \hspace{1cm} \text{false}

Since \( a^4 = b^4 \), thus \( a = b \) or \( a = -b \) or \( a = ib \) or \( a = -ib \) \hspace{1cm} \text{true}

“Since \( a^2 = b^2 \), thus \( a = b \)” is a false statement because the relation defined by the condition \( y = x^2 \) is a binary mapping, which is not an injection nor surjection.

Proper classification allows for evaluation of a logical statement (true or false) without solving appropriate arithmetic equations.
Definitions and notations

Logical statement:
any statement that could be characterized as true or false

Connectives between the logical statements:

→ “if…then” (x > 5) → (x^2 > 25)
notice: (x^2 > 25) → (x > 5) is not a correct statement

⇒ Logical imply, “if…then” statement which is always correct

↔ “if and only if,” “equivalent to” (x > 5) ↔ (2x > 10)
by definition: a ↔ b means (a → b) ∧ (b → a)

⇔ “equivalent to” statement which is always correct

∧ “and” (x^2 > 25) ∧ (x > 0) → (x > 5)

∨ “or” (x^2 > 25) ↔ (x > 5) ∨ (x < -5)

¬ “not” ¬(x > 5) ↔ (x < 5) ∨ (x = 5) ↔ (x ≤ 5)

Quantifiers:

∀ “for all,” “for any” ∀x∈R (x^2 ≥ 0 )

∃ “there exists at least one” ∀δ>0 ∃ε (0 < ε < δ )

∃! “there exists one and only one” ∀x≠0 ∃! y (x·y = y·x = 1)
Set Notations

**Set:** “any collection of elements (or members) defined as belonging to the set”

There are sets and there are elements of sets. A set contains elements (or members) of the set. Elements (members) belong to a set.

- ∈ “belongs to”  \( \pi \in \mathbb{R} \)
- \( \notin \) “does not belong to”  \( \pi \notin \mathbb{N} \)
- \( \ni \) “contains”  \( \mathbb{R} \ni \pi \)
- \( \not\ni \) “does not contain”  \( \mathbb{N} \not\ni \pi \)
- \( \subseteq \) “is a subset of”  \( \mathbb{N} \subseteq \mathbb{R} \)
- \( \subset \) “is a subset of, or equal to”  \( A \subset B \leftrightarrow (A \subseteq B) \vee (A = B) \)
- \( \ni \) “contains a subset of”  \( \mathbb{R} \ni \mathbb{N} \)
- \( \ni \ni \) “contains a subset of, or equal to”  \( A \ni \ni B \leftrightarrow (A \ni B) \vee (A = B) \)
- \( \cup \) “union of sets”  \( \mathbb{N} = \{1, 2, 3, \ldots\}, \mathbb{N}^* = \{-1, -2, -3, \ldots\}, \mathbb{I} = (\mathbb{N} \cup \{0\}) \cup \mathbb{N}^* \)
- \( \cap \) “intersection of sets”  \( \mathbb{N} \cap \mathbb{N}^* = \emptyset \)
- \( \emptyset \) “empty set”  \( \forall A \emptyset \subseteq A \)
Specifications of Sets

How to define a set?

1) List all the elements of the set: \( X = \{ x_1, x_2, x_3, x_4, \ldots x_n \} \)

2) Indicate the distinctive attribute of the elements \( X = \{ x \mid (x \in \mathbb{R}) \land (x^2 - x > 0) \} \)

Basic sets of numbers

The set of natural numbers \( \mathbb{N} = \{ 1, 2, 3, 4, \ldots \} \)

The set of integer numbers \( \mathbb{I} = \{ \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \} \)

The set of prime numbers \( \mathbb{P} = \{ p \mid (p \in \mathbb{N}) \land (p = m \cdot n) \land (m, n \in \mathbb{N}) \Leftrightarrow (m=1) \lor (n=1) \} \)

The set of rational numbers \( \mathbb{Q} = \{ q \mid (\exists m, n \in \mathbb{I}) \land (n \neq 0) \land (q = m/n) \} \)

The set of real numbers \( \mathbb{R} \) (rigorous definition is quite bulky, e.g. Dedekind’s cut of a numerical axis)

A closed interval \([r_1, r_2] = \{ r \mid (r \in \mathbb{R}) \land (r_1 \leq r \leq r_2) \} \)

An open interval \((r_1, r_2) = \{ r \mid (r \in \mathbb{R}) \land (r_1 < r < r_2) \} \)
Set Axioms

1) The Axiom of Extension (or set equality): \( (A = B) \iff (\forall x \in A \rightarrow x \in B) \land (\forall x \in B \rightarrow x \in A) \)

\[ A = \{2, 4, 6\}, \; B = \{x \mid x/2 \in \mathbb{N} \land x < 8\}, \; A = B \]

2) The Containment Axiom (subset existence): \( A \subset B \iff \forall x \; (x \in A \rightarrow x \in B) \), 

Example \( \mathbb{N} \subset \mathbb{I} \subset \mathbb{R} \)

Proper subset \( (A \subset B) \land (A \neq B) \)

Consequences:

2.1) Corollary: \( \forall X \; (X \subset X) \)

2.2) Equivalency of sets: \( (A = B) \iff ((A \subset B) \land (B \subset A)) \)

3) The Axiom of Union: \( \forall (A,B) \; \exists C \; ((x \in C) \iff (x \in A) \lor (x \in B)) \)

\[ C = A \cup B \]

4) The Axiom of Intersection: \( \forall (A,B) \; \exists C \; ((x \in C) \iff (x \in A) \land (x \in B)) \)

\[ C = A \cap B \]

5) The empty set: \( \emptyset \) - a specific set that has only one subset \( \emptyset \subset \emptyset \)

Property of \( \emptyset \): it has no elements.

\[ \forall X \; (\emptyset \subset X) \]

Uniqueness of the empty set \( \emptyset = \emptyset' \)

6) Power Set Axiom: \( \forall A \; \exists P(A) \) (called “Power Set of the set A”) \( \forall p \in P(A) \; \exists S_A \subset A \; (p = S_A) \) (“elements of which are all subsets of A”)
∈ and ⊆ : big difference?

\[ A = \{a_1, a_2, a_3\} \quad B = \{b_1, b_2, b_3, b_4\} \]

Consider \( C = \{A, B\} \)

How many elements are in \( C \)? – Just two: \( A \) and \( B \).

Is it correct that \( a_1 \in C \)? – No. \( A \in C \) and \( B \in C \), that’s it.

\( a_1 \in A \) and \( A \in C \), but \( a_1 \not\in C \)

Furthermore, \( A \in C \), but \( A \not\subset C \). Instead, \( \{A\} \subset C \)

Consider \( D = A \cup B \)

How many elements are in \( D \)? – 7: all \( a_i \) and all \( b_j \).

\( \{a_1\} \subset A \) and \( A \subset D \), thus \( \{a_1\} \subset D \)
Proof for uniqueness of the empty set

Assume contrary: there are two different empty sets $\emptyset \neq \emptyset'$

Then, as long as empty set is a subset of any set $\emptyset \subset \emptyset'$ and $\emptyset' \subset \emptyset$

Using the equivalency of sets criterion from the containment axiom, $\emptyset = \emptyset'$

Notice, however, that

$\emptyset$ is an empty set, a set that has zero elements in it.
Analogy: an empty folder [set] that contains no files [elements]

$\{\emptyset\}$ is a set that contain one element: $\emptyset$.
Analogy: a folder [set] that contains one element, which is an empty folder [empty set]

The analogy is not complete, though: different empty folders are different, but there is only one unique empty set
Example of the Power Set

A = \{a, b, c\}

P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}

\[
\begin{array}{cccc}
1 & 3 & 3 & 1 \\
C_3^0 &= 1 & C_3^1 &= 3 & C_3^2 &= 3 & C_3^3 &= 1 \\
\end{array}
\]

Total number of elements in the power set N = 2^n (if n is finite)

Number of combinations

\[
C_n^m = \frac{n!}{m!(n-m)!} = \text{Total number of permutations of n elements} \\
\text{Total number of permutations of m elements)(Number of permutations of all other elements)}
\]

(choose m elements out of n possible, order does not matter; number of permutations for a set of p elements is p!)

\[
N = \sum_{m=0}^{n} C_n^m \quad \text{Compare with} \quad (x + y)^n = \sum_{m=0}^{n} C_n^m x^m y^{n-m} \quad N = (1+1)^n = 2^n
\]
Graphical presentation of sets

The plane represents all elements of all sets (this is as weird as it sounds)

\[ x \not\in A \]
\[ \sim(x \in A) \]
Or simply
\[ \sim A \]

\[ x \in A \]
Or simply
\[ A \]
Subset

\[ B \subseteq A \]
Intersection of sets

\[ C = A \cap B \]
Union of sets

$A \cup B$

$A$

$B$

$A \cup B$
Examples


given 

\[(A \subseteq C) \rightarrow \forall B \ (A \cap B) \subseteq C\]

\[((A \subseteq C) \land (B \subseteq C)) \rightarrow (A \cup B) \subseteq C\]

Diagram is an illustration only

Rigorous proof still has to rely on the axioms and definitions:

E.g. \(\forall x \in A \cap B \rightarrow x \in A, \ A \subseteq C \rightarrow x \in C\)
Sample Problems

#1) \[ A = \{ x \mid (x \in \mathbb{N}) \land x^2 < 10 \} \]
\[ B = \{ x \mid (x \in \mathbb{R}) \land x^2 > 4 \} \]
find \( A \cap B \) and \( A \cup B \)

\[ A = \{ x \mid (x \in \mathbb{N}) \land x^2 < 10 \} = \{1, 2, 3\} \]
\[ B = \{ x \mid (x \in \mathbb{R}) \land x^2 > 4 \} = \{ x, (x \in \mathbb{R}) \land ((x > 2) \lor (x < -2)) \} \]

\[ A \cap B = \{3\} \]

\[ A \cup B = \{1, x \mid (x \in \mathbb{R}) \land ((x \geq 2) \lor (x < -2)) \} \]
Sample Problems

#2) \( A = \{a, b\} \)
\( B = \{c, d\} \)

find \( P(A \cap B) \) and \( P(A \cup B) \)

\[
P(A \cap B) = P(\emptyset) = \{\emptyset\}
\]

\[
P(A \cup B) = P(\{a, b, c, d\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}
\]
Sample Problems

#3) Sets A and B are such that $A \cap B = \emptyset$. Number of elements in sets A and B are $n_a$ and $n_b$ respectively. Is it always correct that number of elements in the set $C = A \cup B$ is equal to $n_a + n_b$?

Count all elements in C that also belong to A. The number is $n_a$. Then count all elements in C that also belong to B. There are $n_b$ such elements. There are no other elements in C. Also, each element in C is counted exactly once because $A \cap B = \emptyset$.

Thus $n_c = n_a + n_b$. 
Sample Problems

#4) Is this a correct statement?

\[(A \cap B = A) \land (A \cap C = C) \rightarrow (C \subset B)\]

\[
(A \cap C = C) \Rightarrow \forall x \in C \ x \in A \cap C \Rightarrow x \in A
\]

\[
(A \cap B = A) \Rightarrow \forall x \in A \ x \in A \cap B \Rightarrow x \in B
\]

\[
(\forall x \in C \ x \in A) \land (\forall x \in A \ x \in B) \iff (C \subset B)
\]

Yes, the statement is correct
Sample Problems

5) Is this a correct statement?

\[(A \cup B) \cap (B \cup C) = B\]

Consider an element \(x\): \((x \in A) \land (x \in C) \land (x \notin B)\).

\[(x \in A) \Rightarrow (x \in A \cup B)\]
\[(x \in C) \Rightarrow (x \in B \cup C)\]

\[x \in A \cup B \land x \in B \cup C \Rightarrow x \in (A \cup B) \cap (B \cup C).\)

On the other hand, \((x \notin B)\) if such elements may exist, the statement must be considered wrong.

\[(A \cup B) \cap (B \cup C) = B\]
\[(A \cup B) \cap (B \cup C) \neq B\]
Exercises

1) \( A = \{ x \mid (x \in \mathbb{I}) \land x^2 < 10 \} \)
   \( B = \{ x \mid (x \in \mathbb{R}) \land x > 0 \} \)
   find \( A \cap B \) and \( A \cup B \)

2) \( A = \{ a, b, c \} \)
   \( B = \{ b, c, d \} \)
   find \( P(A \cap B) \) and \( P(A \cup B) \)

3) Number of elements in sets \( A, B, \) and \( C = (A \cap B) \) are \( n_a, n_b, \) and \( n_c \) respectively.
   Find number of elements in \( D = A \cup B \)

4) Is this a correct statement?
   \(( (A \cap B = A) \land (A \cap C = A) ) \rightarrow (A \subseteq B \cap C )\)

5) Is this a correct statement?
   \(( (A \cap B) \cap (B \cap C) = \emptyset ) \rightarrow (A \cap C = \emptyset )\)