Towards absolute invariants of images under translation, rotation, and dilation

Feng Lin
Department of Electrical and Computer Engineering, Wayne State University, Detroit, MI 48202, USA

Robert D. Brandt
Image Processing Systems Division, Environmental Research Institute of Michigan, Ann Arbor, MI 48107-8618, USA

Received 6 November 1991

Abstract


We propose several types of phase invariants for images. The phase Taylor invariant and the phase Hessian invariant are invariant under translation. The phase Mellin invariant is invariant under rotation and dilation. We also derive invariants under translation and rotation. All the invariants obtained are absolute in the sense that they preserve all information except that of position, orientation or scale.

Keywords. Invariants, Fourier transform, Fourier–Mellin transform, translation, rotation, dilation.

1. Introduction

Unlike many computations which computers can perform much faster and more accurately than human beings, most of the computations required for vision and pattern recognition applications cannot be performed very efficiently by computers. For example, a human being can easily identify two-dimensional shapes regardless of their position, orientation and scale. In order for computers to do the same thing, either a search must be made, comparing models with different positions, orientations and scales, or one needs to extract features which are invariant under changes in position, orientation and scale. The latter potentially avoids the many computations required to carry out searches. Many different types of invariant features and invariant representations have been studied. Moments have been used to describe the mass (intensity) distributions of images (Teh and Chin (1988)). Fourier descriptors have been used to approximate the boundaries of shapes (Flannery and Horner (1989)). The magnitude of the Fourier transform has been used as a translation invariant (Castro and Morandi (1987)), and the magnitude of the Fourier–Mellin transform has been used as an invariant of rotation (Sheng (1989)). Despite these and other studies, the underlying
mathematical issues in the image invariant problem need to be examined more thoroughly. Furthermore, most invariants proposed in the literature are only relative invariants because information other than that of position, orientation and scale is also discarded.

The purpose of this paper is two-fold: (i) to establish a mathematical foundation for studying various invariants of images; and (ii) to derive absolute invariants under translation, rotation and dilation. The word 'absolute' refers to the fact that those invariants retain all information except that of position, orientation and scale. Our approach is a frequency domain approach. We construct invariants from the Fourier transform or the Fourier–Mellin transform of an image. Thus, we call our invariants 'phase' invariants. Some preliminary results of this approach have been presented in Brandt and Lin (1990) and Brandt et al. (1990).

The first invariant proposed in this paper is the phase Taylor invariant, which is a modification of the Fourier transform of an image. It yields a representation which uniquely characterizes the image up to its position. The representation retains the rotational symmetry property of the Fourier transformation (that is, rotations in the spatial plane correspond to rotations in the frequency plane), and is invariant under translation. It is known that the magnitude spectrum exhibits the properties of invariance under translation and rational symmetry. But the phase spectrum, while satisfying the rotational symmetry property, has a linear part which depends on the position of the image with respect to the planar coordinate system. We can calculate this linear part. The phase Taylor invariant can then be obtained by removing this linear part from the Fourier transform of the image. To relate the phase Taylor invariant with moments, we will show that the coefficients of the linear part represent 'the center of mass'.

Another way to remove the linear part of the phase is to differentiate it twice. This leads to the phase Hessian invariant. The phase Hessian invariant is invariant under translation and is less sensitive to non-constant background and multiple objects but is not rotationally symmetric. Since the property of rotational symmetry is important in deriving invariants under translation and rotation, we modify the phase Hessian invariant. The modified phase Hessian invariant retains the merits of the phase Hessian invariant and is rotationally symmetric.

To derive invariants under rotation and dilation, we use the Fourier–Mellin transform of an image, which is essentially the Fourier transform of the image with respect to the logarithmic-polar coordinates. A change in orientation or scale will result in a linear shift in the phase of its Fourier–Mellin transform. Therefore, we can construct an invariant under rotation and dilation in a way similar to that of constructing the phase Hessian invariant. The construction involves taking second-order differences of the phase of the Fourier–Mellin transform. The invariant obtained is called the phase Mellin invariant.

Since the phase Taylor invariant and the modified phase Hessian invariant are rotationally symmetric, we can construct invariants under translation and rotation from their Fourier–Mellin transforms.

The treatment in this paper is rather theoretical. We believe that implementation issues, though very important, are separated issues. We have implemented and simulated some of the invariants derived in this paper. The results of simulations will be reported separately.

2. Invariants under translation

2.1. The phase Taylor invariant

Abstractly, an image is represented by a function \( f(x, y) \geq 0 \) in the spatial \((x, y)\)-plane. The translated image of \( f(x, y) \) is

\[
S_{(\alpha, \beta)}(f(x, y)) := f(x + \alpha, y + \beta).
\]

370
The rotated image of \( f(x, y) \) is
\[
R_{\theta}(f(x, y)) := f(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta).
\]
The dilated image of \( f(x, y) \) is
\[
D_{q}(f(x, y)) := f(qx, qy).
\]
An invariant \( I(\omega_x, \omega_y) \) of \( f(x, y) \) is a transformation of \( f(x, y) \) that is uniquely determined by \( f(x, y) \):
\[
I(\omega_x, \omega_y) = \mathcal{A}[f(x, y)].
\]

**Definition 1.** \( I(\omega_x, \omega_y) \) is **invariant** under translation if for all \( \alpha \) and \( \beta \),
\[
\mathcal{A}[f(x, y)] = \mathcal{A}[S_{(\alpha, \beta)}(f(x, y))].
\]
That is, \( I(\omega_x, \omega_y) \) is invariant under translation if translation in the spatial \((x, y)\)-plane does not affect \( I(\omega_x, \omega_y) \).

**Definition 2.** \( I(\omega_x, \omega_y) \) is an **absolute invariant** under translation if it is invariant and
\[
\mathcal{A}[f_1(x, y)] = \mathcal{A}[f_2(x, y)] \Rightarrow (\exists \alpha, \beta) f_1(x, y) = S_{(\alpha, \beta)}(f_2(x, y)).
\]
That is, \( I(\omega_x, \omega_y) \) is an absolute invariant under translation if it uniquely characterizes an image up to its position.

Similarly, we can define invariants and absolute invariants under rotation and dilation.

An important property of invariants is the property of rotational symmetry defined as follows.

**Definition 3.** \( I(\omega_x, \omega_y) \) is **rotationally symmetric** if for all \( \theta \),
\[
\mathcal{A}[R_{\theta}(f(x, y))] = R_{\theta}(\mathcal{A}[f(x, y)]).
\]
That is, \( I(\omega_x, \omega_y) \) is rotationally symmetric if rotations in the spatial \((x, y)\)-plane correspond to rotations in the frequency \((\omega_x, \omega_y)\)-plane.

To study invariants, we begin with the Fourier transform of \( f(x, y) \):
\[
F(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i(x\omega_x + y\omega_y)} \, dx \, dy.
\]
Denote it by
\[
F(\omega_x, \omega_y) = A(\omega_x, \omega_y) e^{-i\psi(\omega_x, \omega_y)}
\]
where \( A(\omega_x, \omega_y) \) is the magnitude (spectrum) and \( \psi(\omega_x, \omega_y) \) is the phase (spectrum).

It is well known and easy to show that \( F(\omega_x, \omega_y) \), and hence \( A(\omega_x, \omega_y) \) and \( \psi(\omega_x, \omega_y) \) are rotationally symmetric. It is also easy to show that the magnitude, \( A(\omega_x, \omega_y) \), is invariant under translation. However, the phase \( \psi(\omega_x, \omega_y) \) is not invariant under translation. To obtain an absolute invariant under translation, we modify \( F(\omega_x, \omega_y) \) to the phase Taylor invariant as presented in the following theorem.

**Theorem 1.** Suppose \( f(x, y) \) is an integrable nonnegative function and its Fourier transform \( F(\omega_x, \omega_y) \) is differentiable at the origin. Then the following complex function, called the phase Taylor invariant,
\[
T(\omega_x, \omega_y) = F(\omega_x, \omega_y) e^{-i(a\omega_x + b\omega_y)},
\]

where
\[
a = -i \frac{|F(0,0)|}{F(0,0)} \frac{\partial}{\partial \omega_x} \frac{F(\omega_x, \omega_y)}{|F(\omega_x, \omega_y)|} (0,0) \quad \text{and} \quad b = -i \frac{|F(0,0)|}{F(0,0)} \frac{\partial}{\partial \omega_y} \frac{F(\omega_x, \omega_y)}{|F(\omega_x, \omega_y)|} (0,0),
\]
satisfies the properties of invariance under translation and rotational symmetry.

Proof. First, observe that \(F(0,0) = 0\) only if \(f(x,y) = 0\) almost everywhere, since \(f(x,y) \geq 0\) by assumption. In the case \(f(x,y) = 0\), the function \(T(\omega_x, \omega_y)\), is identically zero regardless of our definition of \(a\) and \(b\). Now if \(F(0,0) \neq 0\), then differentiability of \(F\) implies differentiability of \(F/|F|\). Therefore, the phase Taylor invariant is well defined.

Now we show that the phase Taylor invariant is invariant under translation. That is, if \(T_1(\omega_x, \omega_y)\) is the phase Taylor invariant of \(f_1(x,y)\) and \(T_2(\omega_x, \omega_y)\) is the phase Taylor invariant of \(f_2(x,y) = S(\alpha, \beta)(f_1(x,y)) = f_1(x + \alpha, y + \beta)\), then
\[
T_1(\omega_x, \omega_y) = T_2(\omega_x, \omega_y).
\]
The Fourier transforms of \(f_1(x,y)\) and \(f_2(x,y)\) are related by
\[
F_2(\omega_x, \omega_y) = F_1(\omega_x, \omega_y) e^{i(\alpha \omega_x + \beta \omega_y)}.
\]
It can be calculated that
\[
a_2 = a_1 + \alpha \quad \text{and} \quad b_2 = b_1 + \beta.
\]
Therefore,
\[
T_2(\omega_x, \omega_y) = F_2(\omega_x, \omega_y) e^{-i(\alpha \omega_x + \beta \omega_y)} = T_1(\omega_x, \omega_y).
\]

Next we show that the phase Taylor invariant is rotationally symmetric. That is, if \(T_1(\omega_x, \omega_y)\) is the phase Taylor invariant of \(f_1(x,y)\) and \(T_2(\omega_x, \omega_y)\) is the phase Taylor invariant of \(f_3(x,y) = R_{\theta}(f_1(x,y)) = f_1(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)\), then
\[
T_3(\omega_x, \omega_y) = R_{\theta}(T_1(\omega_x, \omega_y)) = T_1(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta).
\]
The Fourier transforms of \(f_1(x,y)\) and \(f_2(x,y)\) are related by
\[
F_3(\omega_x, \omega_y) = F_1(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta).
\]
It can be calculated that
\[
a_3 = a_1 \cos \theta - b_1 \sin \theta \quad \text{and} \quad b_3 = a_1 \sin \theta + b_1 \cos \theta.
\]
Therefore,
\[
T_3(\omega_x, \omega_y) = F_3(\omega_x, \omega_y) e^{-i(\alpha \omega_x + \beta \omega_y)} = T_1(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta). \quad \square
\]

The idea behind Theorem 1 is that in order to get an invariant under translation, we have to eliminate the linear part of the phase. The resulting alteration of the Fourier transform corresponds to translating the original image by \((a,b)\). In the next theorem we recognize that this particular translation results in normalizing the image so that its 'center of mass' is located at the origin.

Theorem 2. The phase Taylor invariant is the Fourier transform of the image with the origin of the coordinate system located at the center of mass (intensity) of the image.

Proof. Since multiplying the Fourier transform of an image by \(e^{i(a \omega_x + b \omega_y)}\) is equivalent to moving the image by \((a,b)\) (see the proof of Theorem 1), we need only to show that
\[
a = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y) \, dx \, dy, \quad b = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x,y) \, dx \, dy.
\]
where $a$ and $b$ are defined in Theorem 1. We prove the first equation. The second equation can be proved in a similar manner.

Denote by $F_1(\omega_x, \omega_y)$ and $F_i(\omega_x, \omega_y)$ the real and imaginary parts of $F(\omega_x, \omega_y)$ respectively. That is

$$F(\omega_x, \omega_y) = F_1(\omega_x, \omega_y) + iF_i(\omega_x, \omega_y).$$

It can be shown that

$$F_1(0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy, \quad \frac{\partial F_1(0, 0)}{\partial \omega_x} = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) \, dx \, dy.$$

Now

$$a = -i \frac{|F(0, 0)|}{F(0, 0)} \frac{\partial}{\partial \omega_x} \frac{F_1(\omega_x, \omega_y)}{|F(\omega_x, \omega_y)|}(0, 0) = \frac{1}{F_i(0, 0)} \frac{\partial F_i(0, 0)}{\partial \omega_x} = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{x f(x, y) \, dx \, dy}{f(\omega_x, \omega_y)}.$$

The phase Taylor invariant is a complex function. Its magnitude is $A(\omega_x, \omega_y)$. Denote its phase by $\phi(\omega_x, \omega_y)$. Then we can represent the phase Taylor invariant by a vector

$$\mathcal{T}(\omega_x, \omega_y) = \begin{bmatrix} A(\omega_x, \omega_y) \\ \phi(\omega_x, \omega_y) \end{bmatrix} = : \mathcal{T}[f(x, y)].$$

From the above discussion, we conclude:

Corollary 1. The phase Taylor invariant $\mathcal{T}(\omega_x, \omega_y)$ is an absolute invariant under translation and is rotationally symmetric.

Theorems 1 and 2 say that the phase Taylor invariant can be calculated either by subtracting the linear part from the phase of the Fourier transform, or by taking the Fourier transform of the centroid-normalized images. This clearly establishes the relationship between the dominant spatial and frequency domain approaches to the image invariant problem.

2.2. The phase Hessian invariant

Another way of removing the linear part of the phase is to differentiate it twice. Since there are two independent variables, $\omega_x$ and $\omega_y$, there are four possible ways of doing this. We will use subscripts to denote partial derivatives, so for example $\psi_{\omega_x \omega_y}(\omega_x, \omega_y)$ denotes the second derivative of $\psi(\omega_x, \omega_y)$ with respect to $\omega_x$. The matrix of second partial derivatives, called the Hessian, is symmetric as long as the phase function is sufficiently smooth. We will assume this to be the case and obtain another invariant as follows.

Theorem 3. Suppose $f(x, y)$ is an integrable nonnegative function and its Fourier transform $F(\omega_x, \omega_y)$ is twice differentiable. Then the following functions, called the Hessian invariant,

(i) $A(\omega_x, \omega_y)$,  
(ii) $\psi_{\omega_x \omega_y}(\omega_x, \omega_y)$, 
(iii) $\psi_{\omega_x \omega_x}(\omega_x, \omega_y)$,  
(iv) $\psi_{\omega_y \omega_y}(\omega_x, \omega_y)$

satisfy the property of invariance under translation.

Proof. The fact that $F(\omega_x, \omega_y)$ is twice differentiable and $f(x, y) \geq 0$ implies that $\psi(\omega_x, \omega_y)$ is twice differentiable. Therefore, the phase Hessian invariant is well defined.

To show that it is invariant under translation, we need to show that for $f_2(x, y) = S_{(\alpha, \beta)}(f_1(x, y)) = f_1(x + \alpha, y + \beta),$

$$A_1(\omega_x, \omega_y) = A_2(\omega_x, \omega_y), \quad \psi_{\omega_x \omega_x}(\omega_x, \omega_y) = \psi_{\omega_x \omega_x}(\omega_x, \omega_y),$$

where $a$ and $b$ are defined in Theorem 1. We prove the first equation. The second equation can be proved in a similar manner.
\[ \psi_{1\omega_x,\omega_y}(\omega_x, \omega_y) = \psi_{2\omega_x,\omega_y}(\omega_x, \omega_y), \quad \psi_{1\omega_x,\omega_y}(\omega_x, \omega_y) = \psi_{2\omega_x,\omega_y}(\omega_x, \omega_y) \]

where the subscript \( i \) denotes the invariant for \( f_i(x, y) \).

The above is indeed true because (see the proof of Theorem 1)

\[ A_1(\omega_x, \omega_y) = A_2(\omega_x, \omega_y) \quad \text{and} \quad \psi_1(\omega_x, \omega_y) + a\omega_x + b\omega_y = \psi_2(\omega_x, \omega_y). \]

The advantage of the phase Hessian invariant is that, since there is no longer a dependence on the calculation of the image centroid, the nature of the sensitivity to nonconstant background and multiple objects is more of the nature of the sensitivity of the magnitude spectrum alone to those factors. While this sensitivity has not been eliminated, we believe it has been substantially reduced.

The disadvantage of the phase Hessian invariant is that the representation is not rotationally symmetric. To overcome this disadvantage, we will modify the phase Hessian invariant. But first, let us prove that the phase Hessian invariant is absolute. To do this, we show that the phase Taylor invariant can be recovered from the phase Hessian invariant as follows.

**Theorem 4.** Let \( \phi(\omega_x, \omega_y) \) be the unique function satisfying

\begin{align*}
(1) & \quad \phi_{\omega_x,\omega_x}(\omega_x, \omega_y) = \psi_{\omega_x,\omega_x}(\omega_x, \omega_y), \\
(2) & \quad \phi_{\omega_y,\omega_y}(\omega_x, \omega_y) = \psi_{\omega_y,\omega_y}(\omega_x, \omega_y), \\
(3) & \quad \phi_{\omega_x,\omega_y}(\omega_x, \omega_y) = \psi_{\omega_x,\omega_y}(\omega_x, \omega_y), \\
(4) & \quad \phi_x(0, 0) = 0, \\
(5) & \quad \phi_y(0, 0) = 0, \\
(6) & \quad \phi(0, 0) = 0.
\end{align*}

Then

\[ T(\omega_x, \omega_y) = A(\omega_x, \omega_y) e^{-i\phi(\omega_x, \omega_y)}. \]

**Proof.** Clearly, \( \phi(\omega_x, \omega_y) \) is unique. Since

\[ T(\omega_x, \omega_y) = F(\omega_x, \omega_y) e^{-i(\omega_x + b\omega_y)} = A(\omega_x, \omega_y) e^{-i\psi(\omega_x, \omega_y)} e^{-i(\omega_x + b\omega_y)}, \]

we only need to prove that

\[ D(\omega_x, \omega_y) := \psi(\omega_x, \omega_y) + a\omega_x + b\omega_y - \phi(\omega_x, \omega_y) = 0. \]

By (1), (2) and (3),

\[ D_{\omega_x,\omega_x}(\omega_x, \omega_y) = D_{\omega_y,\omega_y}(\omega_x, \omega_y) = D_{\omega_x,\omega_y}(\omega_x, \omega_y) = 0. \]

This implies that

\[ D(\omega_x, \omega_y) = C_1 + C_2 \omega_x + C_3 \omega_y \]

for some constants \( C_1, C_2 \) and \( C_3 \). We want to show that

\[ C_1 = C_2 = C_3 = 0. \]

Note that

\[ a = -i \frac{|F(0, 0)|}{F(0, 0)} \frac{\partial}{\partial \omega_x} \frac{F(\omega_x, \omega_y)}{|F(\omega_x, \omega_y)|} (0, 0) = -\psi_{\omega_x}(0, 0) \]

and

\[ b = -i \frac{|F(0, 0)|}{F(0, 0)} \frac{\partial}{\partial \omega_y} \frac{F(\omega_x, \omega_y)}{|F(\omega_x, \omega_y)|} (0, 0) = -\psi_{\omega_y}(0, 0). \]

Therefore, by (4),

\[ C_2 = D_{\omega_x}(0, 0) = \psi_{\omega_x}(0, 0) + a - \phi_{\omega_x}(0, 0) = 0. \]

Similarly, by (5),

\[ C_3 = D_{\omega_y}(0, 0) = \psi_{\omega_y}(0, 0) + b - \phi_{\omega_y}(0, 0) = 0. \]
Finally, by (6) and the fact that \( \psi(0,0) = 0 \),
\[
C_1 = D(0,0) = \psi(0,0) - \phi(0,0) = 0.
\]
\[\square\]

The following example illustrates how to find \( \phi(\omega_x, \omega_y) \).

Example 1. If
\[
\psi_{\omega_x \omega_x}(\omega_x, \omega_y) = 2\omega_y, \quad \psi_{\omega_y \omega_y}(\omega_x, \omega_y) = 0, \quad \psi_{\omega_x \omega_y}(\omega_x, \omega_y) = 2\omega_x
\]
then we can find \( \phi(\omega_x, \omega_y) \) by solving the equations (1)–(6) in Theorem 4 as
\[
\phi(\omega_x, \omega_y) = \omega_x^2 \omega_y.
\]

Let us denote the phase Hessian invariant by a vector
\[
H(\omega_x, \omega_y) = \begin{bmatrix}
A(\omega_x, \omega_y) \\
\psi_{\omega_x \omega_x}(\omega_x, \omega_y) \\
\psi_{\omega_y \omega_y}(\omega_x, \omega_y) \\
\psi_{\omega_x \omega_y}(\omega_x, \omega_y)
\end{bmatrix}
\]
Then from the above discussion, we conclude

**Corollary 2.** The phase Hessian invariant \( H(\omega_x, \omega_y) \) is an absolute invariant under translation.

2.3. The modified phase Hessian invariant

It is not difficult to show that the phase Hessian invariant \( H(\omega_x, \omega_y) \) is not rotationally symmetric. This will cause a problem when we construct an invariant under translation and rotation (to be discussed in Section 4). To regain the property of rotational symmetry we need to modify the phase Hessian invariant as follows.

**Theorem 5.** Suppose \( f(x,y) \) is an integrable nonnegative function and its Fourier transform \( F(\omega_x, \omega_y) \) is twice differentiable. Then the following functions, called the modified phase Hessian invariant,

(i) \( A(\omega_x, \omega_y) \),
(ii) \( I_1(\omega_x, \omega_y) = \psi_{\omega_x \omega_x} + \psi_{\omega_y \omega_y} \),
(iii) \( I_2(\omega_x, \omega_y) = \psi_{\omega_x \omega_x} + \psi_{\omega_y \omega_y} + \psi_{\omega_x \omega_y} \omega_x \omega_y \),
(iv) \( I_3(\omega_x, \omega_y) = -\omega_y^2 + \omega_x \omega_y \psi_{\omega_x \omega_y} \omega_x \omega_y + (\omega_x^2 + \omega_x \omega_y) \psi_{\omega_x \omega_y} \omega_x \omega_y \)

satisfy the properties of invariance under translation and rotational symmetry.

**Proof.** The modified phase Hessian invariant is well defined and invariant under translation, because the phase Hessian invariant is. We show that it is rotationally symmetric as follows.

Let \( F_1(\omega_x, \omega_y) \) be the Fourier transform of \( f_1(x,y) \). The Fourier transform of \( f_3(x,y) = R_\theta(f_1(x,y)) = f_1(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta) \) is (see the proof of Theorem 1)
\[
F_3(\omega_x, \omega_y) = R_\theta(F_1(\omega_x, \omega_y)) = F_1(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta).
\]
In other words,

(i) \( A_3(\omega_x, \omega_y) = A_1(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta) \)

and

\[ \psi_3(\omega_x, \omega_y) = \psi_1(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta). \]

Therefore,

\[ \psi_{3 \omega_x \omega_y}(\omega_x, \omega_y) = (\cos \theta)^2 \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta) \]

\[ + (\sin \theta)^2 \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta) \]

\[ - 2 \sin \theta \cos \theta \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta), \]

\[ \psi_{3 \omega_x \omega_y}(\omega_x, \omega_y) = (\sin \theta)^2 \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta) \]

\[ + (\cos \theta)^2 \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta) \]

\[ + 2 \sin \theta \cos \theta \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta), \]

\[ \psi_{3 \omega_x \omega_y}(\omega_x, \omega_y) = \psi \theta \cos \theta \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta) \]

\[ - \sin \theta \cos \theta \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta) \]

\[ + (\cos \theta)^2 - (\sin \theta)^2 \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta). \]

Hence,

(ii) \( \psi_{3 \omega_x \omega_y}(\omega_x, \omega_y) + \psi_{3 \omega_x \omega_y}(\omega_x, \omega_y) = \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta) \]

\[ + \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta), \]

(iii) \( \psi_{3 \omega_x \omega_y}^2(\omega_x, \omega_y) + \psi_{3 \omega_x \omega_y}^2(\omega_x, \omega_y) + \psi_{3 \omega_x \omega_y}(\omega_x, \omega_y) \psi_{3 \omega_x \omega_y}(\omega_x, \omega_y) \)

\[ = \psi_{1 \omega_x \omega_y}^2 \omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta \]

\[ + \psi_{1 \omega_x \omega_y}^2 \omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta \]

\[ + \psi_{1 \omega_x \omega_y}^2 \omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta \]

\[ \times \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta), \]

(iv) \( \psi_{3 \omega_x \omega_y}(\omega_x, \omega_y) + \psi_{3 \omega_x \omega_y}(\omega_x, \omega_y) \psi_{3 \omega_x \omega_y}(\omega_x, \omega_y) \psi_{3 \omega_x \omega_y}(\omega_x, \omega_y) \)

\[ = \psi_{1 \omega_x \omega_y}^2 \omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta \]

\[ + \psi_{1 \omega_x \omega_y}^2 \omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta \]

\[ + \psi_{1 \omega_x \omega_y}^2 \omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta \]

\[ \times \psi_{1 \omega_x \omega_y}(\omega_x \cos \theta + \omega_y \sin \theta, -\omega_x \sin \theta + \omega_y \cos \theta). \]

That is, the modified phase Hessian invariant is rotationally symmetric. \( \Box \)
We can recover the phase Hessian invariant from the modified phase Hessian invariant as shown in the following example.

Example 2. If $I_1(\omega_x, \omega_y)$, $I_2(\omega_x, \omega_y)$ and $I_3(\omega_x, \omega_y)$ are given as

$$I_1(\omega_x, \omega_y) = 2\omega_y, \quad I_2(\omega_x, \omega_y) = 4(\omega_x^2 + \omega_y^2), \quad I_3(\omega_x, \omega_y) = 2\omega_x^4 + 4\omega_x^2\omega_y - 4\omega_x\omega_y^2 - 2\omega_y^3$$

then we can solve for $\psi_{\omega_x,\omega_y}(\omega_x, \omega_y)$, $\psi_{\omega_x,\omega_y}(\omega_x, \omega_y)$ and $\psi_{\omega_x,\omega_y}(\omega_x, \omega_y)$. There are two solutions, one does not satisfy the properties of Hessian. Therefore the phase Hessian invariant is uniquely given as

$$\psi_{\omega_x,\omega_y}(\omega_x, \omega_y) = 2\omega_y, \quad \psi_{\omega_x,\omega_y}(\omega_x, \omega_y) = 0, \quad \psi_{\omega_x,\omega_y}(\omega_x, \omega_y) = 2\omega_x.$$

Let us denote the modified phase Hessian invariant by a vector

$$\mathcal{H}(\omega_x, \omega_y) = \begin{bmatrix} A(\omega_x, \omega_y) \\ I_1(\omega_x, \omega_y) \\ I_2(\omega_x, \omega_y) \\ I_3(\omega_x, \omega_y) \end{bmatrix} =: \mathcal{H}[f(x, y)].$$

Then from the above discussion, we conclude:

**Corollary 3.** The modified phase Hessian invariant $\mathcal{H}(\omega_x, \omega_y)$ is an absolute invariant under translation and is rotationally symmetric.

3. Invariants under rotation and dilation

We now consider the problem of finding a representation for images which is invariant under rotation and dilation. We use the Fourier–Mellin transform of an image, which is essentially the Fourier transform of the image with respect to the logarithmic–polar coordinates, $(r, t)$. Denote

$$\tilde{f}(r, t) := f(e^t \sin t, e^t \cos t).$$

Then the rotated image of $\tilde{f}(r, t)$ is

$$R_\theta(\tilde{f}(r, t)) = \tilde{f}(r, t + \theta)$$

and the dilated image of $\tilde{f}(r, t)$ is

$$D_\phi(\tilde{f}(r, t)) = \tilde{f}(r + \ln \phi, t).$$

The Fourier–Mellin transform of $\tilde{f}(r, t)$ is

$$\tilde{F}(\omega, k) = \int_{-\infty}^{\infty} \int_0^{2\pi} \tilde{f}(r, t) e^{-i(kt + \omega r)} \, dt \, dr = \bar{A}(\omega, k) e^{-i\psi(\omega, k)}$$

where $\bar{A}(\omega, k)$ is the magnitude and $\psi(\omega, k)$ is the phase.

To obtain an absolute invariant under rotation and dilation, we need to remove the linear part of the phase. We can do this in a way similar to that in Section 2.2. Since the Fourier–Mellin transform $\tilde{F}(\omega, k)$ is discrete in $k$, we will take the difference instead of the differentiation of the phase. Denote

$$\psi_{\omega k}(\omega, k) = \psi_{\omega}(\omega, k) - \psi_{\omega}(\omega, k - 1) \quad \text{and} \quad \psi_{kk} = \psi_{\omega}(\omega, k) - 2\psi_{\omega}(\omega, k - 1) + \psi_{\omega}(\omega, k - 2).$$

Then we have the following theorem.
Theorem 6. Suppose \( \tilde{f}(r, t) \) is an integrable nonnegative function and its Fourier–Mellin transform \( \tilde{F}(\omega, k) \) is twice differentiable. Then the following functions, called the Mellin invariant,

(i) \( \tilde{A}(\omega_x, \omega_y) \), (ii) \( \tilde{\psi}_{\omega \omega}(\omega, k) \), (iii) \( \tilde{\psi}_{kk}(\omega, k) \), (iv) \( \tilde{\psi}_{\omega k}(\omega, k) \)

satisfy the property of invariance under rotation and dilation.

**Proof.** The fact that \( \tilde{F}(\omega, k) \) is twice differentiable and \( \tilde{f}(r, t) \geq 0 \) implies that \( \tilde{\psi}(\omega, k) \) is twice differentiable. Therefore, the phase Mellin invariant is well defined.

Let \( \tilde{F}_1(\omega, k) \) be the Fourier–Mellin transform of \( \tilde{f}(r, t) \). Then we can show that the Fourier–Mellin transform of \( \tilde{f}_2(r, t) = D_\phi(R_\theta(\tilde{f}_1(r, t))) = \tilde{f}_1(r + \ln \varphi, t + \theta) \) is

\[
\tilde{F}_2(\omega, k) = \tilde{F}_1(\omega, k) e^{i(\ln \varphi \omega + \theta k)}.
\]

Therefore,

\[
\tilde{A}_2(\omega, k) = \tilde{A}_1(\omega, k) \quad \text{and} \quad \tilde{\psi}_2(\omega, k) = \tilde{\psi}_1(\omega, k) - (\ln \varphi \omega + \theta k).
\]

Hence,

\[
\tilde{\psi}_{2\omega\omega}(\omega, k) = \tilde{\psi}_{1\omega\omega}(\omega, k), \quad \tilde{\psi}_{2\omega k}(\omega, k) = \tilde{\psi}_{1\omega k}(\omega, k), \quad \tilde{\psi}_{2kk}(\omega, k) = \tilde{\psi}_{1kk}(\omega, k).
\]

This proves that the phase Mellin invariant is invariant under rotation and dilation. \( \square \)

Let us denote the phase Mellin invariant by a vector

\[
M(\omega, k) = \begin{bmatrix}
\tilde{A}(\omega, k) \\
\tilde{\psi}_{\omega\omega}(\omega, k) \\
\tilde{\psi}_{\omega k}(\omega, k) \\
\tilde{\psi}_{kk}(\omega, k)
\end{bmatrix} =: \mathcal{M}[\tilde{f}(r, t)].
\]

We can prove the following theorem.

**Theorem 7.** The phase Mellin invariant \( M(\omega, k) \) is an absolute invariant under rotation and dilation.

**Proof.** Let \( \tilde{A}_1(\omega, k) e^{i\tilde{\psi}_1(\omega, k)} \) and \( \tilde{A}_2(\omega, k) e^{i\tilde{\psi}_2(\omega, k)} \) be the Fourier–Mellin transform of \( \tilde{f}_1(r, t) \) and \( \tilde{f}_2(r, t) \) respectively. To prove that the phase Mellin invariant is an absolute invariant under rotation and dilation, we only need to show that if

\[
\tilde{A}_1(\omega, k) = \tilde{A}_2(\omega, k), \quad \tilde{\psi}_{2\omega\omega}(\omega, k) = \tilde{\psi}_{1\omega\omega}(\omega, k),
\]

\[
\tilde{\psi}_{2\omega k}(\omega, k) = \tilde{\psi}_{1\omega k}(\omega, k), \quad \tilde{\psi}_{2kk}(\omega, k) = \tilde{\psi}_{1kk}(\omega, k)
\]

then

\[
\tilde{f}_2(r, t) = \tilde{f}_1(r + C_1, t + C_2)
\]

for some constants \( C_1 \) and \( C_2 \). This is true because by the hypothesis and the fact that \( \tilde{\psi}_2(0, 0) = \tilde{\psi}_1(0, 0) = 0 \) we have that

\[
\tilde{\psi}_2(\omega, k) = \tilde{\psi}_1(\omega, k) + C_1 \omega + C_2 k.
\]

\( \square \)

4. **Invariants under translation and rotation**

Because the phase Taylor invariant and the modified phase Hessian invariant are rotationally symmetric, we can construct invariants under translation and rotation by modifying the Fourier–Mellin transforms of the phase Taylor invariant or the modified phase Hessian invariant.

In the following notations, when applying \( \mathcal{M}[\cdot] \) to a vector, we apply it to each component of that vector.
Theorem 8. The following invariant, called the phase Taylor–Mellin invariant,
\[ M[T[f(x, y)]] \]
if it exists, is an absolute invariant under translation and rotation.

Proof. The phase Taylor–Mellin invariant is invariant under translation and rotation because
\[ M[R_\theta(S_{\alpha, \beta})(f(x, y))] = M[R_\theta(T[S_{\alpha, \beta}](f(x, y)))] \]
(rotational symmetry)
\[ = M[R_\theta(T[f(x, y)])] \] (invariant under translation)
\[ = M[T[f(x, y)]] \] (invariant under rotation).

The phase Taylor–Mellin invariant is absolute because both the phase Taylor invariant and the phase Mellin invariant are absolute. □

Similarly, we can prove

Theorem 9. The following invariant, called the phase Hessian–Mellin invariant,
\[ M[H[f(x, y)]] \]
if it exists, is an absolute invariant under translation and rotation.

Acknowledgement

This research is supported in part by the National Science Foundation under grant ECS-9008947. The authors would like to acknowledge the work of Gong Liu on implementation of the invariants reported here.

References