An optimal control approach to robust control design

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We propose an optimal control approach to robust control design. Our goal is to design a state feedback to stabilize a system under uncertainty. We translate this robust control problem into an optimal control problem of minimizing a cost. Because the uncertainty bound is reflected in the cost, the solution to the optimal control problem is a solution to the robust control problem. Our approach can deal with both linear and non-linear systems. Furthermore it can handle both matched and unmatched uncertainties. It can also handle uncertainty in the control input matrix.

1. Introduction

We say that a controller is robust if it works even if the actual system deviates from its ideal model on which the controller design is based. Needless to say, it is very important for a controller to be robust, because, as modern day systems become more and more complex, it is in fact impossible to find the exact model of a system. First, despite the great scientific progresses made so far, there are still phenomena that are poorly understood and hence impossible to model precisely. Second, some modern systems are so complex that even if accurate models are available, they will be too complicated to use. Finally, some systems may be subject to deterioration or other changes during their lifetime. All these facts indicate the necessity of having robust controllers for such systems.

Because of its importance, robust control has been studied extensively, and is reflected in the literature. Most related to this paper is the work in Leitmann (1979), where robust stabilization of linear systems was studied in a state space setting. It was shown that robust stabilization can be achieved by a state feedback based on a properly chosen Lyapunov function, which is independent of uncertainties, if the so-called matching condition is satisfied. The matching condition requires that the uncertainty in the nominal state space model be in the range of the nominal input matrix. This approach was generalized in Barmish et al. (1983), Barmish (1985), Petersen and Hollot (1986), Khargonekar et al. (1990), Zak (1990), etc. and is called the quadratic stability or stabilizability. It was later shown (Olbror and Cieslik 1988, Swi and Corless 1991) that the matching condition is also necessary for robust quadratic stabilization if an arbitrarily fast exponential decay is required.

There are, of course, other approaches to robust control. Among them, the $H_{\infty}$ approach (Zames 1981) has been well developed, and several books (Francis 1987, Green and Limebeer 1995, Zhou 1996) on the subject are available. Another approach to parametric uncertainties was introduced in Kharitnov (1978) and subsequently developed in Kraus et al. (1987), Petersen (1987), Barmish (1989), Fu et al. (1989), etc.

The approach to be presented in this book is quite different from all the above. We do not attempt to solve a robust control problem directly. Rather, we translate the robust control problem into an optimal control problem. After the translation, we can erase the robust control problem from our memory and concentrate on the optimal control problem. As long as we can solve the optimal control problem, the robust control problem is guaranteed to be solved. In fact, the solution to the optimal control problem is a solution to the robust control problem if the matching condition is satisfied (otherwise, a computable sufficient condition needs to be checked and part of the solution to the optimal control problem is a solution to the robust control problem).

For linear systems, the optimal control problem often reduces to a linear quadratic regulator (LQR) problem, whose solution can always be obtained by solving an algebraic Riccati equation. Although the Riccati equation is also used in other approaches to robust control, this fact alone by no means implies that the approaches are equivalent.

In fact, our optimal control approach is unique in the following senses: (1) It is conceptually very simple. Only very limited background is needed to understand the approach. (2) It can be applied to both linear and non-linear systems essentially in the same way. Unlike some other approaches to robust control, there is no conceptual barrier that will prevent the optimal control approach from being used in non-linear systems. In Lin and Zhang (1994a), we have shown how this approach can be applied to aircraft hovering control, a non-linear and non-minimum phase system, where an analytic solution of non-linear state feedback is obtained. Even if the optimal control problem cannot be solved analytically for non-linear systems, it can be solved numerically.

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The paper is organized as follows. In §2 we will review the result of Lin et al. (1992), where the robust control problem for systems with matched uncertainties is discussed. In §3, we will study systems with unmatched uncertainties. In §4 we will consider uncertainties in the input matrix. In §5 we will discuss unmatched input matrix uncertainties. A numerical example will be given in §6.

2. Preliminary

Consider the non-linear system

$$\dot{x} = A(x) + B(x)u + B(x)f(x)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. $B(x)f(x)$ models the uncertainty in the system dynamics. Since the uncertainty is in the range of $B(x)$, the matching condition is satisfied. We will assume $f(0) = 0$ and $A(0) = 0$ so that $x = 0$ is an equilibrium.† In this paper, stability is always with respect to $x = 0$. Our goal is to solve the following.

Robust control problem: Find a feedback control law $u = u_0(x)$ such that the closed-loop system

$$\dot{x} = A(x) + B(x)u_0(x) + B(x)f(x)$$

is globally asymptotically stable for all uncertainties $f(x)$ satisfying the condition that there exists a non-negative function $f_{\text{max}}(x)$ such that

$$\|f(x)\| \leq f_{\text{max}}(x)$$

We can translate this robust control problem into the following.

Optimal control problem: For the nominal system

$$\dot{x} = A(x) + B(x)u$$

find a feedback control law $u = u_0(x)$ that minimizes the cost functional

$$\int_0^\infty (f_{\text{max}}(x)^2 + x^T x + u^T u) \, dt$$

The relation between the robust control problem and the optimal control problem is obtained in Lin et al. (1992) and shown in the following.

Theorem 1: If the solution to the optimal control problem exists, then it is a solution to the robust control problem.

By this theorem, we can solve the robust control problem by solving the optimal control problem. Methods to solve optimal control problems can be found in, for example, Bryson and Ho (1975), Sage and White (1977), Lewis and Syrmos (1995).

† It will be the only equilibrium if the robust control problem is solvable.

3. Unmatched uncertainty

In this paper, we assume that uncertainty is not in the range of $B(x)$. Consider the non-linear system

$$\dot{x} = A(x) + B(x)u + C(x)f(x)$$

where $C(x)$ is a matrix of dimension $n \times p$ and $C(x) \neq B(x)$. Again, we will assume $f(0) = 0$ and $A(0) = 0$ so that $x = 0$ is an equilibrium. We need to solve the following.

Robust control problem: Find a feedback control law $u = u_0(x)$ such that the closed-loop system

$$\dot{x} = A(x) + B(x)u_0(x) + C(x)f(x)$$

is globally asymptotically stable for all uncertainties $f(x)$ satisfying the following conditions.

(1) There exists a non-negative function $g_{\text{max}}(x)$ such that

$$\|f(x)\| \leq g_{\text{max}}(x)$$

(2) There exists a non-negative function $f_{\text{max}}(x)$ such that

$$\|B(x)^+ C(x)f(x)\| \leq f_{\text{max}}(x)$$

where $^+$ denotes the (Moore–Penrose) pseudo-inverse.

Obviously, the first assumption implies the second. However, by introducing both $g_{\text{max}}(x)$ and $f_{\text{max}}(x)$, we can use the least restrictive bound.

We decompose the uncertainty $C(x)f(x)$ into the sum of a matched component and an unmatched component by projecting $C(x)f(x)$ onto the range of $B(x)$. Thus, we take

$$C(x)f(x) = B(x)B(x)^+ C(x)f(x)$$

$$+ (I - B(x)B(x)^+) C(x)f(x)$$

This robust control problem will be translated into the following.

Optimal control problem: For the auxiliary system

$$\dot{x} = A(x) + B(x)u + (I - B(x)B(x)^+) C(x)v$$

find a feedback control law $(u_0(x), v_0(x))$ that minimizes the cost functional
\[ \int_0^\infty (f_{\text{max}}(x)^2 + \rho^2 g_{\text{max}}(x)^2 + \beta^2 \|x\|^2 + \|u\|^2 + \rho^2 \|v\|^2) \, dt \]

where \( \rho \) and \( \beta \) are some (positive) constants that serve as design parameters.

The relationship between the robust control problem and the optimal control problem is given in the following.

**Theorem 2:** If one can choose \( \rho \) and \( \beta \) such that the solution to the optimal control problem, denoted by \((u_0(x), v_0(x))\), exists and the following condition is satisfied

\[ 2\rho^2\|v_0(x)\|^2 \leq \beta^2\|x\|^2, \quad \forall x \in \mathbb{R}^n \quad (1) \]

for some \( \beta' \) such that \( |\beta'| < |\beta| \), then \( u_0(x) \), the u-component of the solution to the optimal control problem, is a solution to the robust control problem.

**Proof:** To simplify the notations, in this proof we will eliminate, if appropriate, the explicit reference to \( x \) when we denote a function of \( x \). Let \((u_0, v_0)\) be the solution to the optimal control problem. Define

\[ V(x_0) = \min_{u \in \mathbb{R}^n, v \in \mathbb{R}^n} \int_0^\infty (f_{\text{max}}(x)^2 + \rho^2 g_{\text{max}}(x)^2 + \beta^2 \|x\|^2 + \|u\|^2 + \rho^2 \|v\|^2) \, dt \]

to be the minimum cost of the optimal control from some initial state \( x_0 \). The Hamilton–Jacobi–Bellman equation gives us

\[ \min_{u \in \mathbb{R}^n, v \in \mathbb{R}^n} (f_{\text{max}}^2 + \rho^2 g_{\text{max}}^2 + \beta^2 \|x\|^2 + \|u\|^2 + \rho^2 \|v\|^2) \]

\[ + V_x^T(A + Bu + (I - BB^+)Cv) = 0 \]

Therefore, \((u_0, v_0)\) satisfy

\[ f_{\text{max}}^2 + \rho^2 g_{\text{max}}^2 + \beta^2 \|x\|^2 + \|u_0\|^2 + \rho^2 \|v_0\|^2 \]

\[ + V_x^T(A + Bu_0 + (I - BB^+)Cv_0) = 0 \quad (2) \]

\[ 2u_0 + V_x^TB = 0 \quad (3) \]

\[ 2\rho^2v_0 + V_x^T(I - BB^+)C = 0 \quad (4) \]

We now show that \( u_0 \) is a solution to the robust control problem, i.e. the equilibrium \( x = 0 \) of

\[ \dot{x} = A + Bu_0 + Cf \]

is globally asymptotically stable for all admissible uncertainties \( f \). To do this, we show that \( V(x) \) is a Lyapunov function. Clearly

\[ V(x) > 0, \quad x \neq 0 \]

\[ V(x) = 0, \quad x = 0 \]

We can show \( \dot{V}(x) < 0 \) for \( x \neq 0 \). In fact, by (2), (3) and (4)

\[ \dot{V}(x) = V_x^T(A + Bu_0 + Cf) \]

\[ = V_x^T(A + Bu_0 + (I - BB^+)Cv_0 + BB^+Cf) \]

\[ + (I - BB^+)C(f - v_0) \]

\[ = V_x^T(A + Bu_0 + (I - BB^+)Cv_0) + V_x^TB^+Cf \]

\[ + V_x^T(I - BB^+)C(f - v_0) \]

\[ = -f_{\text{max}}^2 - \rho^2 g_{\text{max}}^2 - \beta^2 \|x\|^2 - \|u_0\|^2 - \rho^2 \|v_0\|^2 \]

\[ - 2u_0^TB^+Cf - 2\rho^2v_0^T(f - v_0) \]

Since

\[ -\|u_0\|^2 - 2u_0^TB^+Cf = -\|u_0 + B^+Cf\|^2 + \|B^+Cf\|^2 \]

\[ - 2\rho^2v_0^Tf \leq \beta^2(\|v_0\|^2 + \|f\|^2) \]

we then have, using (1)

\[ \dot{V}(x) \leq -(f_{\text{max}}^2 - \|B^+Cf\|^2) - \rho^2(g_{\text{max}}^2 - \|f\|^2) \]

\[ - \|u_0 + B^+Cf\|^2 + 2\rho^2\|v_0\|^2 - \beta^2 \|x\|^2 \]

\[ \leq 2\rho^2\|v_0\|^2 - \beta^2 \|x\|^2 \]

\[ = 2\rho^2\|v_0\|^2 - \beta^2 \|x\|^2 - (\beta^2 - \beta'^2) \|x\|^2 \]

\[ \leq -(\beta^2 - \beta'^2) \|x\|^2 < 0 \]

Thus, the conditions of the Lyapunov stability theorem are satisfied. Consequently, there exists a neighbourhood of 0, \( \mathcal{N} = \{x: \|x\| \leq c\} \) for some \( c > 0 \) such that if \( x(t) \) enters \( \mathcal{N} \), then

\[ x(t) \to 0, \quad \text{as } t \to \infty \]

But \( x(t) \) cannot remain forever outside \( \mathcal{N} \). Otherwise

\[ \|x(t)\| \geq c \]

for all \( t \geq 0 \). Then

\[ V(x(t)) - V(x(0)) = \int_0^t \dot{V}(x(\tau)) \, d\tau \]

\[ \leq -(\beta^2 - \beta'^2) \int_0^t \|x(\tau)\|^2 \, d\tau \]

\[ \leq -(\beta^2 - \beta'^2)c^2t \]

Letting \( t \to \infty \), we have

\[ V(x(t)) \leq V(x(0)) - (\beta^2 - \beta'^2)c^2t \to -\infty \]

which contradicts the fact that \( V(x(t)) \geq 0 \) for all \( x(t) \). Therefore

\[ x(t) \to 0, \quad \text{as } t \to \infty \]
no matter where the trajectory begins. This proves that 
\( u_0 \) is a solution to the robust control problem.

Comparing with the case of matched uncertainty, an additional condition is needed to ensure that the
augmented control (and hence the unmatched uncertainty) is not too large. This condition can be tested after solv-
ing the optimal control problem. This condition is sufficient but not necessary. By proper choice of \( \rho \) and \( \beta \), this sufficient condition can be satisfied in many problems of interest.

4. Uncertainty in the input matrix

We now generalize our optimal control approach to handle uncertainty in the control input matrix.

Consider the non-linear system
\[
\dot{x} = A(x) + B(x)(u + h(x)u) + C(x)f(x)
\]
where \( h(x) \) is an \( m \times m \) uncertainty matrix. The robust control problem is described as follows.

Robust control problem: Find a feedback control law \( u = u_0(x) \) such that the closed-loop system
\[
\dot{x} = A(x) + B(x)(u_0(x) + h(x)u_0(x)) + C(x)f(x)
\]
is globally asymptotically stable for all uncertainties \( f(x) \) and \( h(x) \) satisfying the following conditions.

1. \( h(x) \geq 0 \).
2. There exists a non-negative function \( g_{\text{max}}(x) \)
such that
\[
\|f(x)\| \leq g_{\text{max}}(x)
\]
3. There exists a non-negative function \( f_{\text{max}}(x) \)
such that
\[
\|B(x)^\top C(x)f(x)\| \leq f_{\text{max}}(x)
\]
The last two assumptions are the same as in the previous section and the first assumption is the additional one on the uncertainty in the input matrix. To translate this robust control problem into an optimal control problem, we again decompose the uncertainty \( C(x)f(x) \) as
\[
C(x)f(x) = B(x)B(x)^\top C(x)f(x) + (I - B(x)B(x)^\top)C(x)f(x)
\]
The corresponding optimal control problem is then as follows.

Optimal control problem: For the auxiliary system
\[
\dot{x} = A(x) + B(x)u + (I - B(x)B(x)^\top)C(x)v
\]
find a feedback control law \((u_0(x), v_0(x))\) that minimizes the cost functional
\[
\int_0^\infty (f_{\text{max}}(x)^2 + \rho^2 g_{\text{max}}(x)^2 + \beta^2 \|x\|^2 + \|u\|^2 + \rho^2 \|v\|^2) \, dt
\]
where \( \rho \) and \( \beta \) are (positive) constants which again serve as design parameters.

The relation between the robust control problem and the optimal control problem is shown in the following.

Theorem 3: If one can choose \( \rho \) and \( \beta \) such that the solution to the optimal control problem denoted by \((u_0(x), v_0(x))\) exists and the following condition is satisfied
\[
2\rho^2 \|v_0(x)\|^2 \leq \beta' \|x\|^2,
\]
for some \( \beta' \) such that \( |\beta'| < |\beta| \), then \( u_0(x) \), the u-component of the solution to the optimal control problem, is a solution to the robust control problem.

Proof: As before, we will eliminate, when possible, the explicit reference to \( x \) when we denote a function of \( x \). Assume \((u_0, v_0)\) is the solution to the optimal control problem. Define
\[
V(x_0) = \min_{u \in \mathbb{R}^n, v \in \mathbb{R}^m} \int_0^\infty (f_{\text{max}}(x)^2 + \rho^2 g_{\text{max}}(x)^2 + \beta^2 \|x\|^2 + \|u\|^2 + \rho^2 \|v\|^2) \, dt
\]
to be the minimum cost of the optimal control from some initial state \( x_0 \). The Hamilton–Jacobi–Bellman equation gives us
\[
\min_{u \in \mathbb{R}^n, v \in \mathbb{R}^m} \left( f_{\text{max}}^2 + \rho^2 g_{\text{max}}^2 + \beta^2 \|x\|^2 + \|u\|^2 + \rho^2 \|v\|^2 + V_x^I(A + Bu + (I - BB^\top)Cv) \right) = 0
\]
Therefore, \((u_0, v_0)\) has the properties
\[
f_{\text{max}}^2 + \rho^2 g_{\text{max}}^2 + \beta^2 \|x\|^2 + \|u_0\|^2 + \rho^2 \|v_0\|^2
\]
\[
+ V_x^I(A + Bu_0 + (I - BB^\top)Cv_0) = 0 \quad (5)
\]
\[
2u_0^2 + V_x^I B = 0 \quad (6)
\]
\[
2\rho^2 v_0^2 + V_x^I (I - BB^\top)C = 0 \quad (7)
\]
We now show that \( u_0 \) is a solution to the robust control problem, i.e. the equilibrium \( x = 0 \) of
\[
\dot{x} = A + B(u_0 + hu_0) + Cf
\]
is globally asymptotically stable for all admissible uncertainties \( f \) and \( h \). To do this, we show that \( V(x) \) is a Lyapunov function. Clearly,
\[
V(x) > 0, \quad x \neq 0
\]
\[
V(x) = 0, \quad x = 0
\]
To show \( \dot{V}(x) < 0 \) for \( x \neq 0 \), we have, by (5), (6) and (7).
\[ \dot{V}(x) = V_x^T(A + Bu_0 + Cf + Bh_u_0) \]
\[ = V_x^T(A + Bu_0 + (I - BB^+)Cv_0 + BB^+Cf) \]
\[ + (I - BB^+)C(f - v_0) + Bh_u_0 \]
\[ = V_x^T(A + Bu_0 + (I - BB^+)Cv_0) + V_x^TBB^+Cf \]
\[ + V_x^T(I - BB^+)C(f - v_0) + V_x^TBh_u_0 \]
\[ = -\gamma_2^2 - \rho^2 \gamma_2^2 \beta^2 \|x\|^2 - \|u_0\|^2 - \rho^2 \|v_0\|^2 \]
\[ - 2u_0^TB^+Cf - 2\rho^2 v_0^T(f - v_0) - 2u_0^Th_u_0 \]

Since
\[ -\|u_0\|^2 - 2u_0^TB^+Cf = -\|u_0 + B^+Cf\|^2 + \|B^+Cf\|^2 \]
\[ - 2\rho^2 v_0^Tf \leq \rho^2 \|v_0\|^2 + \|f\|^2 \]
\[ - 2u_0^Th_u_0 \leq 0 \]

we then have
\[ \dot{V}(x) \leq -(\gamma_2^2 - \|B^+Cf\|^2) - \rho^2(\gamma_2^2 - \|f\|^2) \]
\[ + 2\rho^2 \|v_0\|^2 - \beta^2 \|x\|^2 - \|u_0 + B^+Cf\|^2 \]
\[ \leq 2\rho^2 \|v_0\|^2 - \beta^2 \|x\|^2 \]
\[ = 2\rho^2 \|v_0\|^2 - \beta^2 \|x\|^2 - (\beta^2 \|x\|^2 - \beta^2 \|x\|^2) \]
\[ \leq -(\beta^2 - \beta^2) \|x\|^2 < 0 \]

Thus, the conditions of the Lyapunov stability theorem are satisfied. Consequently, there exists a neighbourhood of 0, \( \mathcal{N} = \{x : \|x\| \leq c\} \) for some \( c > 0 \) such that if \( x(t) \) enters \( \mathcal{N} \), then
\[ x(t) \to 0, \quad \text{as } t \to \infty \]

But \( x(t) \) cannot remain forever outside \( \mathcal{N} \). Otherwise,
\[ \|x(t)\| \geq c \]

for all \( t \geq 0 \). Then
\[ V(x(t)) - V(x(0)) = \int_0^t \dot{V}(x(\tau)) d\tau \]
\[ \leq - (\beta^2 - \beta^2) \int_0^t \|x(\tau)\|^2 d\tau \]
\[ \leq - (\beta^2 - \beta^2)c^2 t \]

Letting \( t \to \infty \), we have
\[ V(x(t)) \leq V(x(0)) - (\beta^2 - \beta^2)c^2 t \to -\infty \]

which contradicts the fact that \( V(x(t)) \geq 0 \) for all \( x(t) \). Therefore
\[ x(t) \to 0, \quad \text{as } t \to \infty \]

no matter where the trajectory begins. So we proved that \( u_0(x) \) is a solution to the robust control problem.

Because of the ability of dealing with the uncertainty in the input matrix, we can now apply this optimal control approach to a wider range of systems. One such application is the control of a robot manipulator studied in Lin and Brandt (1996).

5. Unmatched input matrix uncertainty

In the previous section, we assume that the uncertainty in the input matrix is in the range of \( B(x) \). In this section, we will briefly discuss the unmatched input uncertainty. For simplicity, we assume that this is the only uncertainty in the system. That is, we assume that the system is given by
\[ \dot{x} = A(x) + B(x)u + C(x)D(x)u \]

where \( D(x) \) is the uncertainty and \( C(x) \neq B(x) \). As usual, we will assume \( A(0) = 0 \) so that \( x = 0 \) is an equilibrium. We would like to solve the following.

Robust control problems: Find a feedback control law \( u = u_0(x) \) such that the closed-loop system
\[ \dot{x} = A(x) + B(x)u_0(x) + C(x)D(x)u_0(x) \]

is globally asymptotically stable for all uncertainties \( D(x) \) such that \( \|D(x)\| \leq D_{\max}(x) \) for some \( D_{\max}(x) \).

Since \( u \) will be a function of \( x: u = u_0(x) \), we can view \( f(x) = D(x)u_0(x) \) as uncertainty and guess its bound
\[ \|f(x)\| \leq g_{\max}(x) \]
\[ \|B(x)^+C(x)f(x)\| \leq \|B(x)^+C(x)\|g_{\max}(x) \]

Then, using the result in § 3, we will solve the following.

Optimal control problem: For the auxiliary system
\[ \dot{x} = A(x) + B(x)u + (I - B(x)B(x)^+)C(x)v \]

find a feedback control law \( (u_0(x), v_0(x)) \) that minimizes the cost functional
\[ \int_0^\infty \left( \|B(x)^+C(x)\|^2 + \rho^2 \right)g_{\max}(x) \]
\[ + \beta^2 \|x\|^2 + \|u\|^2 + \rho^2 \|v\|^2 \right) dt \]

where \( \rho \) and \( \beta \) are some (positive) constants that serve as design parameters.

It is clear that, by Theorem 2, we have the following.

Theorem 4: If one can choose \( \rho, \beta \) and \( g_{\max}(x) \) such that the solution to the optimal control problem, denoted by \( (u_0(x), v_0(x)) \), exists and the following conditions are satisfied
2\rho^2 \|v_0(x)\|^2 \leq \beta'^2 \|x\|^2

\|D(x)u_0(x)\|^2 \leq g_{\text{max}}(x)^2 \quad \forall x \in \mathbb{R}^n

for such $\beta'$ such that $|\beta'| < |\beta|$, then $u_0(x)$, the $u$-component of the solution to the optimal control problem, is a solution to the robust control problem.

**Proof:** Elementary. \qed

6. Example

We will illustrate the above theoretical results by the following example. Let us consider the non-linear system

\[
\begin{align*}
\dot{x}_1 &= x_2 + p_1 x_1 \cos \left( \frac{1}{x_2 + p_2} \right) + p_3 x_2 \sin (p_4 x_1 x_2) \\
\dot{x}_2 &= u
\end{align*}
\]

where $p_1 \in [-0.2, 0.2]$, $p_2 \in [-10, 100]$, $p_3 \in [0, 0.2]$, and $p_4 \in [-100, 0]$ are the uncertainties.

We can rewrite the state equation as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u +
\begin{bmatrix}
0.2
\end{bmatrix} f(x_1, x_2)
\]

where

\[
f(x_1, x_2) = 5p_1 x_1 \cos \left( \frac{1}{x_2 + p_2} \right) + 5p_3 x_2 \sin (p_4 x_1 x_2)
\]

Therefore

\[
\|f(x_1, x_2)\|^2 \leq x_2^2 + x_2^2 = g_{\text{max}}(x)^2
\]

\[
\|B^+ C f(x_1, x_2)\|^2 = 0 = f_{\text{max}}(x)^2
\]

Clearly

\[
B^+ = (B^T B)^{-1} B^T = B^T =
\begin{bmatrix}
0 & 1
\end{bmatrix}
\]

\[
(I - BB^+)C =
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0.2 \\
0.2
\end{bmatrix} =
\begin{bmatrix}
0 & 0
\end{bmatrix}
\]

Let $\rho = \beta = 1$. Then the corresponding optimal control problem is as follows. For the systems

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u +
\begin{bmatrix}
0.2
\end{bmatrix} v
\]

find a feedback control law $u_0, v_0$ that minimizes the cost functional

\[
\int_0^\infty \left( x^T 
\begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}
x + u^T u + v^T v \right) dt
\]

The solution is given by
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Figure 2. Response of Case 2.

Figure 3. Response of Case 3.
\[ u_0 = -1.2906x_1 - 2.1247x_2 \]

\[ v_0 = -0.5783x_1 - 0.2581x_2 \]

Therefore, the sufficient condition

\[ 2\beta^2 \|v_0\|^2 \leq \beta^2 \|x\|^2 \]

is satisfied and \( u_0 \) is a robust control.

We simulate the closed-loop system with the initial conditions \( x_1(0) = 100, x_2(0) = -50 \). The simulation results for the following four cases are shown in figures 1, 2, 3 and 4 respectively.

Case 1: \( p_1 = -0.2, p_2 = -10, p_3 = 0, p_4 = -100 \)

Case 2: \( p_1 = 0.2, p_2 = 100, p_3 = 0.2, p_4 = 0 \)

Case 3: \( p_1 = 0, p_2 = 0, p_3 = 0, p_4 = 0 \)

Case 4: \( p_1 = -0.2, p_2 = -100, p_3 = -0.2, p_4 = -100 \)

Clearly, the control is robust.

7. Conclusion

Robust control for stabilization of a class of non-linear systems is designed by translating the robust control problem into an optimal control problem with a cost specifically selected to compensate the uncertainty. This approach is applicable to non-linear systems, even if the matching condition is not satisfied. Since in many cases an optimal control problem is easier to solve than a robust control problem, this indirect approach provides an effective alternative to direct approaches studied in the literature.

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References


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