An optimal control approach to robust tracking of linear systems

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In our early work, we show that one way to solve a robust control problem of an uncertain system is to translate the robust control problem into an optimal control problem. If the system is linear, then the optimal control problem becomes a linear quadratic regulator (LQR) problem, which can be solved by solving an algebraic Riccati equation. In this article, we extend the optimal control approach to robust tracking of linear systems. We assume that the control objective is not simply to drive the state to zero but rather to track a non-zero reference signal. We assume that the reference signal to be tracked is a polynomial function of time. We first investigated the tracking problem under the conditions that all state variables are available for feedback and show that the robust tracking problem can be solved by solving an algebraic Riccati equation. Because the state feedback is not always available in practice, we also investigated the output feedback. We show that if we place the poles of the observer sufficiently left of the imaginary axis, the robust tracking problem can be solved. As in the case of the state feedback, the observer and feedback can be obtained by solving two algebraic Riccati equations.

Keywords: robust control; optimal control; LQR problem; tracking problem; observer

1. Introduction

In real life, the systems to be controlled have many uncertainties that can make the performance deviate from the nominal design. So designing effective robust controllers for such systems becomes significant. In the past, there have been many works (Leitmann 1979; Barmish, Corless and Leitmann 1983; Barmish 1985; Okada, Kihara and Furuhata 1985; Petersen and Hollett 1986; Schmitendorf and Barmish 1986; Chen 1988; Beale and Shafai 1989; Stalford and Chao 1989; Dawson, Qu, Lewis and Dorsey 1990; Banda, Yeh and Heise 1991; Faryar and Schmitendorf 1991; Swei and Corless 1991; Tadmor 1991; Han and Chen 1992; Chen 1993; Wang and Zhang 1994; Amato, Pironi and Scala 1996; Battilotti 1997; Benson and Schmitendorf 1997; Chen and Tarng 1997; Ionescu, Oraabreve, Weiss, Oara and Weiss 1999; Shieh, Liang and Mao 2003) on robust control design of uncertain systems. We also contributed to robust control design by proposing an optimal control approach that translates a robust control problem into an optimal control problem (Lin, Brandt and Sun 1992; Lin and Zhang 1994; Lin and Olbrot 1996; Lin and Brandt 1998; Lin, Zhang and Brandt 1999; Lin 2000; Lin 2007). If the system is linear, then the optimal control problem becomes a linear quadratic regulator (LQR) problem, which can be solved by solving an algebraic Riccati equation. In our early work, we focused on the robust regulator problem, that is, the objective of control is to drive the states and outputs to zero. In this article, we investigate the robust tracking problem, that is, the objective of control is to track a non-zero reference signal. We consider two cases: the case based on state feedback and the case using output feedback. For a linear system with the state feedback, the robust tracking problem is similar to the robust regulator problem, which can be solved by solving an algebraic Riccati equation. The case of output feedback is much more complex because we need to design an observer to obtain online state estimation of the states. Nevertheless, we show that, as in the case of state feedback, the observer and feedback can be obtained by solving two algebraic Riccati equations.

The robust tracking problem is also investigated in (Schmitendorf and Barmish 1986; Benson and Schmitendorf 1997; Shieh et al. 2003). Schmitendorf and Barmish (1986) developed a robust tracking method for linear systems with time-invariant uncertainties and constant but unknown disturbances. Shieh et al. (2003) investigated a robust output tracking control problem for a class of uncertain linear systems via a modified optimal linear-quadratic method.

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Benson and Schmitendorf considered a robust observer-based augmented system tracking control problem in Benson and Schmitendorf (1997).

In comparison with the work of (Schmitendorf and Barmish, 1986; Benson and Schmitendorf, 1997; Shieh et al. 2003) for the robust tracking of uncertain systems, our approach is simpler and easier to implement. For example, Schmitendorf and Barmish (1986); Benson and Schmitendorf (1997) solve the robust tracking of uncertain systems by solving some linear matrix inequality (LMI) and there are many parameters which need to be selected. Since no effective procedures are given, it is difficult to select these parameters. The Riccati equation is also used in Shieh et al. (2003) to solve the robust tracking problem. However, it considers only the case that the uncertainty is matched and state feedback is available. We consider both matched and unmatched uncertainties as well as both state and output feedback.

The article is organised as follows: Section 2 reviews the background of robust control design using the optimal control approach. In §3, we consider the design of a state feedback controller for robust tracking of an uncertain system. We consider systems with both matched and unmatched uncertainties. In §4, we investigate the robust tracking problem of uncertain systems using output feedback. Due to the complexity of the problem, we assume that uncertainty is matched in the case of output feedback. Some concluding remarks are given in §5.

2. Background

We propose an optimal control approach to robust control in Lin et al. (1992); Lin and Zhang (1994); Lin and Ozbrot (1996); Lin and Brandt (1998); Lin et al. (1999); Lin (2000) and the results are summarised in a recently published book (Lin 2007). For linear systems, the approach considers the following system

\[ \dot{x} = A(p)x + Bu, \]

where \( x \in \mathbb{R}^n \) are state variables, \( u \in \mathbb{R}^m \) are control inputs, and \( p \in \mathbb{P} \) are uncertain parameters. If the uncertainty of \( A(p) \) is in the range of \( B \), that is, the uncertainty in \( A(p) \) can be written as \( A(p) = A(p_o) + B\phi(p) \) for some \( \phi(p_o) \), where \( p_o \in \mathbb{P} \) is the nominal value of \( p \), then we say that the matching condition is satisfied; otherwise, the matching condition is not satisfied. The optimal control approach considers both matched and unmatched uncertainties. We briefly review the results for matched uncertainties here. For the robust control, the control objective is to solve the following problem.

**Robust control problem:** Find a feedback control law \( u = Kx \) such that the closed-loop system

\[ \dot{x} = A(p)x + Bu = A(p_o)x + Bu + B\phi(p)x \]

is asymptotically stable for any \( p \in \mathbb{P} \).

Our approach is to translate the above robust control problem into an optimal control problem. Since we consider linear systems here, the optimal control problem becomes the following LQR problem.

**Linear quadratic regulator problem:** For the nominal system

\[ \dot{x} = A(p_o)x + Bu, \]

find a feedback control law \( u = Kx \) that minimises the cost functional

\[ \int_0^\infty (x^T Fx + x^T u + u^T u) dt, \]

where \( F \) is an upper bound on the uncertainty \( \phi^T(p)\phi(p) \); that is, for all \( p \in \mathbb{P} \), \( \phi^T(p)\phi(p) \leq F \).

To solve the LQR problem, we first solve the algebraic Riccati equation (note that \( R = R^{-1} = I \))

\[ A(p_o)^T S + S A(p_o) + F + I - SBB^T S = 0 \]

for \( S \). Then the solution to the LQR problem is given by \( u = Kx = -B^T S x \).

The following theorem shows that we can solve the robust control problem by solving the LQR problem.

**Theorem 1** (Lin 2007): The above robust control problem is solvable if the matching condition is satisfied and there exists a nominal value \( p_o \in \mathbb{P} \) such that \( (A(p_o), B) \) is controllable. Furthermore, the solution to the LQR problem is a solution to the robust control problem.

Therefore, solving the robust control problem boils down to solving an algebraic Riccati equation. We will use this idea to solve the robust tracking problem of uncertain systems.

3. Robust tracking using state feedback

In this section, we extend the optimal control approach in another direction, that is, we consider robust tracking of a non-zero reference signal. We consider both matched uncertainty and unmatched uncertainty.

3.1 Matched uncertainty

We first consider the case when the matching condition is satisfied, that is, the system is described by

\[ \dot{x} = A(p_o)x + Bu + B\phi(p)x \]

\[ y = Cx. \]
We want to design a feedback control to ensure that the output $y$ is tracking a reference signal $y_r$. We assume that the reference signal is a polynomial function of time:

$$y_r = a_0 + a_1 t + \cdots + a_{d-1} t^{d-1},$$

where $a_i$, $i=0,1,\ldots,d-1$ are some constant coefficients. Our objective is to design a control law $u$ under which the output asymptotically tracks the reference signal with no error, that is, $y \to y_r$ as $t \to \infty$. In determining the control law, we assume that the following information is available for control: (1) state variables $x$ and (2) integrals of error $e = y - y_r$.

To formalise this, let us define new state variables as

$$\dot{q}_1 = e, \quad \dot{q}_2 = q_1, \quad \cdots \quad \dot{q}_d = q_{d-1}.$$

Clearly, $q_i$ is the $i$th integral of $e$, which is available to the controller. The overall structure of the controlled system is therefore shown in Figure 1.

The state equation with original and new state variables can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_d \end{bmatrix} = \begin{bmatrix} A(p_u) & 0 & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ q_1 \\ q_2 \\ \vdots \\ q_d \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ q_2 \\ \vdots \\ q_d \end{bmatrix} \begin{bmatrix} \phi(p) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u \\ 0 \\ \vdots \\ 0 \end{bmatrix} y_r.$$

Denote the above equation as

$$\dot{z} = A_z(p_u)z + B_z\phi_z(p)z + B_zu + M_1 y_r,$$

where

$$z = \begin{bmatrix} x \\ q_1 \\ \vdots \\ q_d \end{bmatrix}, \quad A_z(p_u) = \begin{bmatrix} A(p_u) & 0 & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B_z = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

Figure 1. Robust tracking with state feedback.

Formally, the robust tracking problem with state feedback can be stated as follows.

Robust tracking problem using state feedback: Find a feedback control law $u = K_z z$ such that the controlled system

$$\dot{x} = A(p_u)x + Bu + B\phi(p)x$$

is asymptotically stable and $y \to y_r$ for all possible $p \in P$.

The upper bound on the uncertainty $\phi_z(p) = [\phi(p) \ 0 \ \cdots \ \ 0]$ is denoted by $F_z$. We further assume that $(A_z(p_u), B_z)$ is controllable, otherwise, the solution may not exist.

To solve the robust tracking problem using state feedback, we first solve the following algebraic Riccati equation

$$A_z^T(p_u)S_z + S_zA_z(p_u) - S_zB_zB_z^T S_z + F_z^T F_z + I = 0.$$

The control for robust tracking can then be obtained as

$$u = K_z z = -B_z^T S_z z.$$

**Theorem 2**: Assume that the matching condition is satisfied and there exists a nominal value $p_u \in P$ such that $(A_z(p_u), B_z)$ is controllable. Then the solution to robust tracking problem using state feedback exists and is given by $u = K_z z = -B_z^T S_z z$, where $S_z$ is the solution to the algebraic Riccati equation

$$A_z^T(p_u)S_z + S_zA_z(p_u) - S_zB_zB_z^T S_z + F_z^T F_z + I = 0.$$
Proof: Since \((A(p), B_2)\) is controllable, the solution to the algebraic Riccati equation exists. We want to show that the closed-loop system
\[
\begin{align*}
\dot{x} &= A(p)x + Bu + B\phi(p)x \\
y &= Cx \\
n\text{is asymptotically stable and } y \to y, \text{ for all possible } p \in P.
\end{align*}
\]
We first show that the system
\[
\dot{z} = A_z(p)z + B_zK_zz
\]

is asymptotically stable for all \(p \in P\). To this end, let us consider the following Lyapunov function candidate:
\[
V(z) = z^TS_zz
\]
Clearly,
\[
V(z) > 0, \quad z \neq 0 \\
V(z) = 0, \quad z = 0.
\]

To show \(V(z) < 0\) for all \(z \neq 0\), we have
\[
V(z) = z^TS_zz + z^TS_z\dot{z}
\]
\[
= (A_z(p)z + B_zK_zz + B_z\phi_z(p)z)^TS_zz
\]
\[
= z^T(A_z(p)S_z + S_zA_z(p) + 2S_zB_zK_z + \phi_z(p)B_z^TS_z + S_zB_z\phi_z(p))z
\]
\[
= z^T(A_z(p)S_z - S_zA_z(p) - 2S_zB_z^TS_z + \phi_z(p)B_z^TS_z + S_zB_z\phi_z(p))z
\]
because
\[
\phi_z(p)B_z^TS_z + S_zB_z\phi_z(p) \leq S_zB_zB_z^TS_z + \phi_z(p)\phi_z(p).
\]
We have
\[
\dot{V}(z) \leq z^T(A_z(p)S_z + S_zA_z(p))
\]
\[
- 2S_zB_zB_z^TS_z + \phi_z(p)\phi_z(p)z
\]
\[
= z^T(A_z(p)S_z + S_zA_z(p))
\]
\[
- 2S_zB_zB_z^TS_z + \phi_z(p)\phi_z(p)z
\]
\[
= -z^TS_z.
\]
In other words,
\[
\dot{V}(z) < 0, \quad z \neq 0 \\
V(z) = 0, \quad z = 0.
\]
By the Lyapunov Stability Theorem, the controlled system is asymptotically stable and
\[
z \to 0 \quad \text{as } t \to \infty.
\]
Let us now consider the extended system
\[
\dot{z} = A_z(p)z + B_zK_zz + My_r.
\]
Taking the \(d\)-th derivative on both sides, we have
\[
z^{(d+1)} = A_z(p)z^{(d)} + B_zK_zz^{(d)}.
\]
This equation has the same form as
\[
\dot{z} = A_z(p)z + B_zK_zz.
\]
Therefore,
\[
z^{(d)} \to 0 \quad \text{as } t \to \infty.
\]
In particular,
\[
q^{(d)}_r = y - y_r \to 0 \quad \text{as } t \to \infty.
\]

Example 1: Let us consider the following second-order system
\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & p \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x
\end{align*}
\]
where \(p \in [0, 10]\) is the uncertainty. We would like to design a state feedback control law to ensure that the output \(y\) is tracking a reference signal \(y_r = 1\) for all \(p \in [0, 10]\). We first construct the augmented system as follows
\[
\dot{z} = A_z(p)z + Bu + My_r,
\]
where
\[
\begin{align*}
z(t) &= \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} \\
A_z(p) &= \begin{bmatrix} A(p) & 0 \\ C & 0 \end{bmatrix} \\
B_z &= \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
M &= \begin{bmatrix} 0 \\ -1 \end{bmatrix}
\end{align*}
\]
Figure 2. Simulation results of Example 1 for $p = 1$.

To solve this problem, let us pick $p = 0$ and verify that $(A_x(p), B_x)$ is controllable. We then calculate $F_z$ as follows.

$$
\Delta A_x(p) = \begin{bmatrix}
\Delta A(p) & 0 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
p & p \\
0 & 0
\end{bmatrix}
$$

$$
= 1 \begin{bmatrix}
p & p \\
0 & 0
\end{bmatrix} = B_x \phi_x(p)
$$

$$
\phi_x^T(p) \phi_x(p) = \begin{bmatrix}
p & p \\
0 & 0
\end{bmatrix} \begin{bmatrix}
p & p \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
p^2 & p^2 \\
p^2 & p^2 \\
0 & 0
\end{bmatrix}
$$

$$
\leq \begin{bmatrix}
100 & 100 & 0 \\
100 & 100 & 0 \\
0 & 0 & 0
\end{bmatrix} = F_z.
$$

Hence, we solve the following algebraic Riccati equation

$$
A_x^T(p) S_z + S_z A_x(p) - S_z B_z B_z^T S_z + F_z + I = 0.
$$

to obtain

$$
S_z = \begin{bmatrix}
23.8461 & 12.1529 & 11.1940 \\
12.1529 & 11.1940 & 1.0000 \\
11.1940 & 1.0000 & 11.1529
\end{bmatrix}
$$

By Theorem 2, the solution to the robust tracking using state feedback exists and is given the following state feedback matrix

$$
K_z = -B_z^T S_z = \begin{bmatrix}
-12.1529 & 11.1940 & 1.0000
\end{bmatrix}
$$

To see the performance of the closed-loop system, we conduct simulations, whose results are shown in Figure 2.

From the simulation results of Figure 2, we see that the system response is not fast. This is because the poles of the controlled system are not far away from the imaginary axis, that is, the stability margin is not large. If we want a large stability margin and fast response, we may want to place the poles of the controlled system at least $\gamma$ distance away from the imaginary axis. In other words, we want to solve the following problem.

**Robust tracking problem with guaranteed stability margin**: For an arbitrary positive real number $\gamma$, find a feedback control law $u = K_z x$ such that the controlled system

$$
\dot{x} = A(p) x + Bu + B\phi(p)x
$$

$$
= A(p) x + BK_z z + B\phi(p)x
$$

$$
y = Cx
$$

has all its poles on the left of $-\gamma$ and $y \to y$, for all possible $p \in P$.

The solution to the above problem is given in the following theorem.

**Theorem 3**: Assume that the matching condition is satisfied and there exists a nominal value $p_0 \in P$ such that $(A_x(p_0), B_x)$ is controllable. Then the solution to robust tracking problem with guaranteed stability margin exists and is given by $u = K_z z = -B_z^T S_z z$, where $S_z$ is the solution to the algebraic Riccati equation

$$
(A_x(p) + B\gamma) S_z + S_z (A_x(p) + B\gamma) - S_z B_z B_z^T S_z + F_z + I = 0.
$$

**Proof**: Since $(A_x(p_0), B_z)$ and hence $(A_x(p_0) + B\gamma, B_z)$ is controllable (for all positive real $\gamma$) the solution to the algebraic Riccati equation exists. By Theorem 2,
we know its solution \( u = K_z z = -B_z^T S_z z \) has the following property: the system

\[
\dot{z} = A_z(p) z + B_z K_z z + \gamma z \\
= A_z(p_0) z + B_z K_z z + B_z \phi_z(p) z + \gamma z
\]

is asymptotically stable and \( y \to y_r \) for all possible \( p \in P \). Denote the real part of complex numbers \( s \) by Re(\( s \)) and the determinant of matrix \( A \) by |\( A \) |

Then we have

\[
(\forall p \in P)(\forall s, \text{Re}(s) \geq 0)|sI - A_z(p_0) - \gamma I - B_z K_z - B_z \phi_z(p)| \neq 0
\]

\[
(\forall p \in P)(\forall s, \text{Re}(s) \geq 0)|(s - \gamma)I - A_z(p_0) - B_z K_z - B_z \phi_z(p)| \neq 0
\]

Let \( s' = s - \gamma \) then \( \text{Re}(s) = \text{Re}(s' + \gamma) = \text{Re}(s') + \gamma \geq 0 \iff \text{Re}(s') \geq -\gamma \). Therefore,

\[
(\forall p \in P)(\forall s', \text{Re}(s') \geq -\gamma)|s'I - A_z(p_0) - B_z K_z - B_z \phi_z(p)| \neq 0
\]

which implies that the controlled system has all its poles on the left of \(-\gamma\). Furthermore, it is clear that \( y \to y_r \) for all possible \( p \in P \).

**Example 2:** Consider the system discussed in Example 1. We would like to design a state feedback control to ensure that the output \( y \) is tracking a reference signal \( y_r \) and has all its poles on the left of \(-2\) (that is, \( \gamma = 2 \)) for all \( p \in [0, 10] \). The augmented system is same as in Example 1, but the algebraic Riccati equation to be solved is different:

\[
(A_z(p_0) + \gamma B_z^T S_z + S_z A_z(p_0) + \gamma I) - S_z B_z B_z^T S_z + F_z I = 0.
\]

The solution is given by

\[
S_z = \begin{bmatrix}
1291.347 & 100.717 & 2338.0459 \\
100.717 & 19.5053 & 150.7905 \\
2338.0459 & 150.7905 & 5684.1965
\end{bmatrix}
\]

By Theorem 3, the feedback matrix is given by

\[
K_z = -B_z^T S_z = -\begin{bmatrix}
100.7170 & 19.5053 & 150.7905
\end{bmatrix}
\]

The simulation results are shown in Figure 3.

Compared with the tracking showed in Figure 2, the tracking in Figure 3 is much faster.

### 3.2 Unmatched uncertainty

We now relax the matching condition, that is, the system is described by

\[
\dot{x} = A(p)x + Bu \\
y = Cx
\]

Our goal is the same as in §3.1. The overall structure of the controlled system is the same as shown in Figure 1. The state equation is then written as

\[
\begin{bmatrix}
\dot{x} \\
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_d
\end{bmatrix} = \begin{bmatrix}
A(p) & 0 & 0 & 0 \\
C & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \cdots & 1 & 0
\end{bmatrix} \begin{bmatrix}
x \\
q_1 \\
q_2 \\
q_d
\end{bmatrix} + \begin{bmatrix}
B & 0 \\
0 & -1 \\
0 & \cdots \\
0 & 0
\end{bmatrix} \begin{bmatrix}
u \\\ny_r
\end{bmatrix}
\]

Denote the above equation as

\[
\dot{z} = A_z(p) z + B_z u + M y_r
\]

where

\[
A_z(p) = \begin{bmatrix}
A(p) & 0 & 0 & 0 \\
C & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \cdots & 1 & 0
\end{bmatrix}
\]

\[
B_z = \begin{bmatrix}
B & 0 \\
0 & -1 \\
0 & \cdots \\
0 & 0
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
0 \\
\cdots \\
0
\end{bmatrix}
\]

![Figure 3. Simulation results of Example 2 for \( p = 1 \).](image-url)
The robust tracking problem with unmatched uncertainty can be formally stated as follows.

**Robust tracking problem with unmatched uncertainty:**
Find a feedback control law \( u = K_zz \) such that the controlled system

\[
\dot{x} = A(p)x + Bu \\
= A(p)x + BK_zz \\
y = Cx
\]

is asymptotically stable and \( y \to y_r \) for all possible \( p \in P \).

In order to solve this robust tracking problem, we first decompose the uncertainty \( \Delta A_z(p) = A_z(p) - A_z(p_0) \) into the sum of a matched component and an unmatched component by projecting it into the range of \( B_z \), that is,

\[
\Delta A_z(p) = B_z^* \Delta A_z(p) + (I - B_zB_z^*) \Delta A_z(p)
\]

where \( B_z^* \) is the pseudo inverse of \( B_z \). We then define \( F_z \) and \( H_z \) as the following upper bounds on the uncertainty

\[
\begin{align*}
\left(B_z^* \Delta A_z(p)\right)^T\left(B_z^* \Delta A_z(p)\right) & \leq F_z \\
\alpha^{-2} \Delta A_z^T(p) \Delta A_z(p) & \leq H_z
\end{align*}
\]

where \( \alpha \geq 0 \) is a design parameter.

To solve the robust tracking problem with state feedback, we first solve the following algebraic Riccati equation

\[
A_z^T(p_0)S_z + S_zA_z(p_0) - S_z\left(B_zB_z^\top + \alpha^2 \rho^{-2}(I - B_zB_z^*)^2\right)S_z \\
+ F_z + \rho^2 H_z + \rho^2 I = 0
\]

where \( \beta > 0 \) and \( \rho > 0 \) are design parameters. The control for robust tracking can then be obtained as \( u = K_zz = -B_z^*S_zz \).

**Theorem 4:** Assume that there exists a nominal value \( p_0 \in \mathbb{P} \) such that \( (A_z(p_0), B_z) \) is controllable. If there exist design parameters \( \alpha > 0, \beta > 0 \) and \( \rho > 0 \) such that the solutions to the following algebraic Riccati equation

\[
A_z^T(p_0)S_z + S_zA_z(p_0) - S_z\left(B_zB_z^\top + \alpha^2 \rho^{-2}(I - B_zB_z^*)^2\right)S_z \\
+ F_z + \rho^2 H_z + \rho^2 I = 0
\]

satisfies

\[
\beta^2 I - 2\alpha^2 \rho^{-2} S_z(I - B_zB_z^*)^T S_z > 0
\]

then the solution to robust tracking problem with unmatched uncertainty exists and is given by

\[
u = K_zz = -B_z^*S_zz.
\]

**Proof:** Since \( (A_z(p_0), B_z) \) is controllable, the solution to the algebraic Riccati equation exists. We want to show that the closed-loop system

\[
\dot{x} = A(p)x + Bu \\
= A(p)x + BK_zz \\
y = Cx
\]

is asymptotically stable and \( y \to y_r \) for all possible \( p \in P \). We first show that the system

\[
\dot{z} = A_z(p)z + B_zK_zz
\]

is asymptotically stable for all \( p \in P \). To this end, let us consider the following Lyapunov function candidate:

\[
V(z) = z^T S_z z
\]

Clearly,

\[
V(z) > 0, \quad z \neq 0 \\
V(z) = 0, \quad z = 0.
\]

To show \( \dot{V}(z) < 0 \) for all \( z \neq 0 \), we have

\[
\dot{V}(z) = z^T S_z \dot{z} + \dot{z}^T S_z z
\]

\[
= (A_z(p)z + B_zK_zz)^T S_z z + z^T S_z (A_z(p)z + B_zK_zz)
\]

\[
= z^T (A_z^T(p)S_z + S_zA_z(p))z - 2z^T S_z B_z B_z^T S_z
\]

\[
= z^T (A_z^T(p_0)S_z + S_zA_z(p_0) + \Delta A_z^T(p)B_z^T B_z^T S_z + \Delta A_z^T(p)(I - B_zB_z^*)^T S_z + S_z B_z B_z^T \Delta A_z(p))
\]

\[
+ z^T \Delta A_z^T(p)B_z^T B_z^T S_z + z^T S_z B_z B_z^T \Delta A_z(p)
\]

\[
+ \left(z^T \Delta A_z^T(p)(I - B_zB_z^*)^T S_zight)
\]

\[
+ z^T S_z(I - B_zB_z^*) \Delta A_z(p) z - z^T S_z B_z B_z^T S_z
\]

Since,

\[
A_z^T(p_0)S_z + S_zA_z(p_0)
\]

\[
- S_z\left(B_zB_z^\top + \alpha^2 \rho^{-2}(I - B_zB_z^*)^2\right)S_z \\
+ F_z + \rho^2 H_z + \rho^2 I = 0
\]

we have

\[
\dot{V}(z) = z^T \left(S_z \left(B_zB_z^\top + \alpha^2 \rho^{-2}(I - B_zB_z^*)^2\right)S_z \\
- F_z - \rho^2 H_z - \rho^2 I\right) z
\]
Furthermore,
\[
- z^T S_z B_z^T S_z z + z^T \Delta A_z^T(p) B_z^T (I - B_z B_z^T) \Delta A_z(p) z
= - z^T \left( - B_z^T S_z + B_z^T \Delta A_z(p) \right) (I - B_z B_z^T) \Delta A_z(p) z
+ z^T (B_z^T \Delta A_z(p) )^T (B_z^T \Delta A_z(p) ) z
\leq z^T \left( B_z^T \Delta A_z(p) \right)^T (B_z^T \Delta A_z(p) ) z
= z^T F_z z
\]
and
\[
- z^T \left( \Delta A_z^T(p) (I - B_z B_z^T)^T S_z + S_z (I - B_z B_z^T) \Delta A_z(p) \right) z
\leq - z^T \alpha^2 \rho^2 S_z (I - B_z B_z^T) (I - B_z B_z^T)^T S_z z
+ z^T \alpha^2 \rho^2 \Delta A_z^T(p) \Delta A_z(p) z
\leq - z^T \alpha^2 \rho^2 S_z (I - B_z B_z^T) (I - B_z B_z^T)^T S_z z + z^T \rho^2 H_z z.
\]

Therefore,
\[
\dot{V}(z) \leq - z^T \left( S_z (B_z^T \alpha^2 + \alpha^2 \rho^2 (I - B_z B_z^T)^2) S_z
- F_z - \rho^2 H_z - \beta^2 I ) z + z^T F_z z
\]
\[
+ z^T \alpha^2 \rho^2 S_z (I - B_z B_z^T) (I - B_z B_z^T)^T S_z
+ z^T \rho^2 H_z z - z^T S_z B_z^T S_z z
\]
\[
= - z^T \left( \beta^2 I - 2 \alpha^2 \rho^2 S_z (I - B_z B_z^T) (I - B_z B_z^T)^T S_z z
\right) z.
\]

If the sufficient condition \( \beta^2 I - 2 \alpha^2 \rho^2 S_z (I - B_z B_z^T) (I - B_z B_z^T)^T S_z > 0 \) is satisfied, then
\[
\dot{V}(z) \leq - z^T \left( \beta^2 I - 2 \alpha^2 \rho^2 S_z (I - B_z B_z^T) (I - B_z B_z^T)^T S_z \right) z < 0.
\]

In other words,
\[
\dot{V}(z) < 0, \quad z \neq 0
\]
\[
\dot{V}(z) = 0, \quad z = 0.
\]

By the Lyapunov Stability Theorem,
\[
z \to 0 \quad \text{as} \quad t \to \infty.
\]

Let us now consider the extended system
\[
\dot{z} = A_z(p) z + B_z K_z z + M_y y.
\]

Taking 4th derivative on both sizes, we have
\[
\dddot{z}(d+1) = A_z(p) \dddot{z}(d) + B_z K_z \dddot{z}(d).
\]

This equation has the same form as
\[
\dddot{z} = A_z(p) \dddot{z} + B_z K_z \dddot{z}.
\]

Therefore,
\[
\dddot{z}(d) \to 0 \quad \text{as} \quad t \to \infty.
\]

In particular,
\[
\dddot{y}_d = y - y, \quad \to 0 \quad \text{as} \quad t \to \infty.
\]

**Example 3:** Consider the following second-order system.
\[
\dot{x} = \begin{bmatrix} p & 10 + p \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]
\[
y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x
\]

where \( p \in [-0.1, 0.1] \) is the uncertainty. We would like to design a feedback control to ensure that the output \( y \) is tracking a reference signal \( y_r \) for all \( p \in [-0.1, 0.1] \). We construct the augmented system as follows.
\[
\dot{z} = A_z(p) z + B_z u + M_y y_r,
\]

where
\[
\begin{align*}
A_z(p) &= \begin{bmatrix} A(p) & 0 \\ 0 & C \end{bmatrix}, \\
B_z &= \begin{bmatrix} B \\ 0 \end{bmatrix}, \\
M &= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.
\end{align*}
\]

To solve this problem, let \( v_0 = 0 \). It is not difficult to check that \( (A_z(p), B_z) \) is controllable. Take \( \alpha = 0.01, \rho = 1, \beta = 10 \), we calculate \( F_z, H_z \) as follows.
\[
\Delta A_z(p) = A_z(p) - A_z(p_0) = \begin{bmatrix} p & 10 + p & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]
\[
\Delta A_z^T(p) B_z^T B_z^T \Delta A_z(p)
= \begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]
\[
\alpha^2 \Delta A_z^T(p) \Delta A_z(p)
= 10^4 \begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = H_z.
\]
Solve the following algebraic Riccati equation

\[ A_z^T(p_o)S_z + S_zA_z(p_o) - S_z \left( B_zB_z^T + \alpha^2 \rho^{-2}(I - B_zB_z^T)^2 \right) S_z + F_z + \rho^2 H_z + \beta^2 I = 0 \]

we obtain

\[
\begin{bmatrix}
25.3050 & 16.7024 & 22.8407 \\
16.7024 & 23.1087 & 9.8757 \\
22.8407 & 9.8757 & 155.4857
\end{bmatrix}
\]

Since

\[
\begin{bmatrix}
99.7675 & -0.1297 & -0.8259 \\
-0.1297 & 99.9247 & -0.3835 \\
-0.8259 & -0.3835 & 95.0602
\end{bmatrix} > 0
\]

by Theorem 4, the state feedback for robust tracking exists and is given by the following feedback matrix

\[
K_z = -B_z^T S_z = \begin{bmatrix} 16.7024 & 23.1087 & 9.8757 \end{bmatrix}
\]

The simulation results are shown in Figure 4.

Similarly, we can solve the robust tracking problem with guaranteed stability margin as follows.

**Theorem 5:** Assume that there exists a nominal value \( p_o \in P \) such that \( (A_z(p_o), B_z) \) is controllable. If there exist design parameters \( \alpha > 0 \), \( \beta > 0 \) and \( \rho > 0 \) such that the solutions to the following algebraic Riccati equation

\[
\begin{align*}
(A_z^T(p_o) + \gamma I)S_z + S_z(A_z(p_o) + \gamma I) \\
- S_z \left( B_zB_z^T + \alpha^2 \rho^{-2}(I - B_zB_z^T)^2 \right) S_z \\
+ F_z + \rho^2 H_z + \beta^2 I = 0
\end{align*}
\]

satisfies

\[
\beta^2 I - 2\alpha^2 \rho^{-2}S_z(I - B_zB_z^T)(I - B_zB_z^T)^T S_z > 0
\]

then the solution to robust tracking problem exists and is given by

\[
u = K_z z = -B_z^T S_z z.
\]

Furthermore, all the poles of the system are on the left of \(-\gamma\).

**Proof:** The proof is similar to that of Theorem 3. \(\square\)

**Example 4:** Consider again the system in Example 3. We would like to design a feedback control to ensure that the output \( y \) is tracking a reference signal \( y_r = 1 \) and has all its poles on the left of \(-1\) for all \( \rho \in [-0.1, 0.1] \). The augmented system

\[
\dot{z} = A_z(p)z + B_zu + My,
\]

is the same as in Example 3. Take \( \alpha = 0.01 \), \( \rho = 1 \), \( \beta = 10 \), \( \gamma = 1 \) as in Example 3 and solve the following algebraic Riccati equation

\[
\begin{align*}
(A_z^T(p_o) + \gamma I)S_z + S_z(A_z(p_o) + \gamma I) \\
- S_z \left( B_zB_z^T + \alpha^2 \rho^{-2}(I - B_zB_z^T)^2 \right) S_z \\
+ F_z + \rho^2 H_z + \beta^2 I = 0
\end{align*}
\]

we obtain

\[
\begin{bmatrix}
37.7248 & 22.0892 & 84.5814 \\
22.0892 & 26.3499 & 34.6476 \\
84.5814 & 34.6476 & 566.6389
\end{bmatrix}
\]

Since

\[
\begin{bmatrix}
98.2847 & -0.7528 & -10.2234 \\
-0.7528 & 99.6623 & -4.3005 \\
-10.2234 & -4.3005 & 34.3531
\end{bmatrix} > 0
\]

by Theorem 5, the state feedback matrix is given

\[
K_z = -B_z^T S_z = \begin{bmatrix} 22.0892 & 26.3499 & 34.6476 \end{bmatrix}
\]

![Figure 4](image-url) Simulation results of Example 3 for \( \rho = 0.1 \).
The simulation results of the controlled system are shown in Figure 5.

Compared with the tracking showed in Figure 4, the tracking in Figure 5 is faster.

Note that robust tracking problem with matched uncertainty is a special case of robust tracking problem with unmatched uncertainty. However, the solution to robust tracking problem with matched uncertainty is not a special case of the solution to robust tracking problem with unmatched uncertainty. Indeed, the solution to robust tracking problem with matched uncertainty is much simpler and always exists, while the solution to robust tracking problem with unmatched uncertainty is more complex and only exists if appropriate design parameters can be found.

4. Robust tracking with output feedback

In this section, we discuss the robust tracking problem when state variables are not directly available for feedback control and hence we need to use observers. In this case, the difficulty in finding a control law increases significantly. Hence, we consider the robust tracking problem under the assumption that the matching condition is satisfied. The system is described by

\[
\dot{x} = A(p_0)x + Bu + B\phi(p)x
\]

\[
y = Cx.
\]

The reference signal to be tracked and the new state variables are the same as in state feedback. In particular,

\[
\dot{z} = A_z(p_0)z + B_z\phi_z(p)z + B_zu + My.
\]

We augment the output equation as

\[
y = C_zz
\]

where \(C_z = [C_z C_z]\) and \(C_z\) is chosen so that the augmented system \((A_z(p_0), C_z)\) is observable. We construct an observer

\[
\dot{\hat{z}} = (A_z(p_0) + LC_z)\hat{z} - Ly_z + B_zu
\]

to estimate state variables \(z\). The feedback is based on

\[
\hat{z} = u = K_z\hat{z}.\]

The overall structure of the controlled system is shown in Figure 6.

Let us define the estimation error as

\[
\bar{z} = z - \hat{z}
\]

Then the state equation for the overall system is as follows.

\[
\dot{\bar{z}} = (A_z(p_0) + B_zK_z)z + B_z\phi_z(p)z - B_zK_z\hat{z} + My
\]

\[
\dot{\hat{z}} = (A_z(p_0) + LC_z)\hat{z} + B_z\phi_z(p)z.
\]

The robust tracking problem to be solved can be stated as follows.

**Robust tracking problem based on observer:** Find a feedback control gain \(K_z\) and an observer gain \(L\) such that the closed-loop system

\[
\dot{\bar{z}} = (A_z(p_0) + B_zK_z)z + B_z\phi_z(p)z - B_zK_z\hat{z} + My
\]

\[
\dot{\hat{z}} = (A_z(p_0) + LC_z)\hat{z} + B_z\phi_z(p)z
\]

is asymptotically stable and \(y \rightarrow y_r\) for any \(p \in P\).

As before, we find \(K_z\) and \(L\) by solving algebraic Riccati equations. In particular, let \(S\) be the solution to the algebraic Riccati equation

\[
A_z^T(p_0)S + SA_z(p_0) - \frac{1}{2}SB_zB_z^TS + G + I = 0
\]

where \(G\) is an upper bound on \(4\phi_z^T(p)\phi_z(p)\); that is, for all \(p \in P\), \(4\phi_z^T(p)\phi_z(p) \leq G\). Let \(P\) be the solution to the algebraic Riccati equation

\[
P(A_z(p_0) + \sigma I)^T + (A_z(p_0) + \sigma I)P - r_nPC_z^TC_z^TP = 0
\]

\[
\text{where} \quad r_n = \frac{\sigma}{2}
\]

\[
B_zB_z^T = 0
\]
where $\sigma$, $r_o$ and $r_c$ are three design parameters to be selected.

After solving the above two algebraic Riccati equations, the gain matrices $K_z$ and $L$ can be obtained as follows:

$$K_z = -B_z^T S$$
$$L = -r_o P C_z^T.$$

**Theorem 6:** Assume that the matching condition is satisfied and there exists a nominal value $p_o \in P$ such that $(A_z(p_o), B_z)$ is controllable and $(A_z(p_o), C_z)$ is observable. If there exist design parameters $\sigma > 0$, $r_o > 0$ and $r_c > 0$ such that the solutions to the following algebraic Riccati equations

$$A_z^T(p_o) S + S A_z(p_o) - \frac{1}{2} S B_z B_z^T S + G + I = 0$$
$$P(A_z(p_o) + \sigma I)^T + (A_z(p_o) + \sigma I) P$$
$$- r_o P C_z^T C_z P + \frac{r_c}{2} B_z B_z^T = 0$$

satisfy

$$-2\sigma P^{-1} - r_o C_z^T C_z + \frac{1}{r_c} S B_z B_z^T S < 0$$

then the solution to robust tracking problem based on observer exists and the gain matrices are given by

$$K_z = -B_z^T S$$
$$L = -r_o P C_z^T.$$

**Proof:** Since $(A_z(p_o), B_z)$ is controllable and $(A_z(p_o), C_z)$ is observable, the solutions to the algebraic Riccati equations exist. Suppose that there exist design parameters $\sigma > 0$, $r_o > 0$ and $r_c > 0$, such that the solution satisfy $-2\sigma P^{-1} - r_o C_z^T C_z + (1/r_c) S B_z B_z^T S < 0$. We first show that the closed-loop system

$$\dot{z} = (A_z(p_o) + B_z K_z) z + B_z \phi_z(p) z - B_z K_z \bar{z}$$
$$\ddot{z} = (A_z(p_o) + L C_z) \bar{z} + B_z \phi_z(p) \bar{z}$$

is asymptotically stable for all $p \in P$. Let us consider the following Lyapunov function candidate:

$$V(z, \bar{z}) = \frac{1}{r_c} z^T S z + \bar{z}^T P^{-1} \bar{z}$$

Clearly,

$$V(z, \bar{z}) > 0, \quad z \neq 0 \lor \bar{z} \neq 0$$
$$V(z, \bar{z}) = 0, \quad z = 0 \lor \bar{z} = 0.$$
To show $\dot{V}(z, \tilde{z}) < 0$ for all $z \neq 0$ and $\tilde{z} \neq 0$, we have

$$\dot{V}(z, \tilde{z}) = \frac{1}{r_c} \tilde{z}^T S z + \frac{1}{r_c} \tilde{z}^T S \tilde{z} + \tilde{z}^T P^{-1} \tilde{z} + \tilde{z}^T P^{-1} \tilde{z}$$

$$= \frac{1}{r_c} ((A_2(p_o) + B_2 K_c) z + B_2 \phi_2(p) z - B_2 K_c \tilde{z}) S z$$

$$+ \frac{1}{r_c} \tilde{z}^T S ((A_2(p_o) + B_2 K_c) z + B_2 \phi_2(p) z - B_2 K_c \tilde{z})$$

$$+ \frac{1}{r_c} \tilde{z}^T P^{-1} ((A_2(p_o) + B_2 K_c) \tilde{z} + B_2 \phi_2(p) \tilde{z})$$

$$= \frac{1}{r_c} \tilde{z}^T (A_2^T(p_o) S + S A_2(p_o)) z$$

$$+ \frac{1}{r_c} \tilde{z}^T (A_2^T(p_o) S + S A_2(p_o)) \tilde{z}$$

$$= \frac{1}{r_c} \tilde{z}^T (A_2^T(p_o) S + S A_2(p_o) + 1) z$$

$$= \frac{1}{r_c} \tilde{z}^T \left( -2 \sigma P^{-1} - r_c C_2^T C_2 + \frac{1}{r_c} S B_2 B_2^T S \right) \tilde{z}$$

$$\leq \frac{1}{r_c} \tilde{z}^T \left( -2 \sigma P^{-1} - r_c C_2^T C_2 + \frac{1}{r_c} S B_2 B_2^T S \right) \tilde{z}.$$

If the sufficient condition $-2 \sigma P^{-1} - r_c C_2^T C_2 + (1/r_c) S B_2 B_2^T S < 0$ is satisfied, then

$$\dot{V}(z, \tilde{z}) \leq \frac{1}{r_c} \tilde{z}^T \left( -2 \sigma P^{-1} - r_c C_2^T C_2 + \frac{1}{r_c} S B_2 B_2^T S \right) \tilde{z}.$$

In other words,

$$\dot{V}(z, \tilde{z}) < 0, \quad z \neq 0 \lor \tilde{z} \neq 0$$

$$\dot{V}(z, \tilde{z}) = 0, \quad z = 0 \land \tilde{z} = 0.$$

Therefore, by the Lyapunov Stability Theorem,

$$z \to 0 \quad \text{as} \quad t \to \infty.$$

Let us now consider the extended system

$$\dot{\tilde{z}} = (A_2(p_o) + B_2 K_c) \tilde{z} + B_2 \phi_2(p) z - B_2 K_c \tilde{z} + M y_r$$

$$\dot{\tilde{z}} = (A_2(p_o) + L C_2) \tilde{z} + B_2 \phi_2(p) z.$$

Taking $d$th derivative on both sides, we have

$$z^{(d+1)} = (A_2(p_o) + B_2 K_c) z^{(d)} + B_2 \phi_2(p) z^{(d)} - B_2 K_c z^{(d)}$$

$$z^{(d+1)} = (A_2(p_o) + L C_2) z^{(d)} + B_2 \phi_2(p) z^{(d)}.$$

This equation has the same form as

$$\dot{z} = (A_2(p_o) + B_2 K_c) z + B_2 \phi_2(p) z - B_2 K_c \tilde{z}$$

$$\dot{\tilde{z}} = (A_2(p_o) + L C_2) \tilde{z} + B_2 \phi_2(p) z.$$

Therefore,

$$z^{(d)} \to 0 \quad \text{as} \quad t \to \infty.$$

In particular,

$$q^{(d)}_y = y - y_r \to 0 \quad \text{as} \quad t \to \infty.$$
Example 5: Consider the following second-order system.

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & -1 + p \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} 1 & 0 \end{bmatrix} x
\]

where \( p \in [-2, 2] \) is the uncertainty. We would like to design an observer-based feedback control to ensure that the output \( y \) is tracking a reference signal \( y_r = 1 \) for all \( p \in [-2, 2] \). We construct the augmented system as follows

\[
\dot{z} = A_z(p)z + B_z u + My_r \\
y_z = C_z z = \begin{bmatrix} C & C_2 \end{bmatrix} z
\]

where

\[
z(t) = \begin{bmatrix} x(t) \\ q(t) \end{bmatrix}, \quad A_z(p) = \begin{bmatrix} A(p) & 0 \\ 0 & C & 0 \end{bmatrix}, \quad B_z = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
M = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

To solve this problem, let \( p_0 = 0, C_2 = 1 \). Then \((A_z(p_0), B_z)\) is controllable and \((A_z(p_0), C_z)\) is observable. \( G \) can be calculated as follows.

\[
\Delta A_z(p) = A_z(p) - A_z(p_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & p & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & p & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = G.
\]

We solve the algebraic Riccati equation

\[
A_z^T(p_0)S + SA_z(p_0) - \frac{1}{2} SB_z B_z^T S + G + I = 0
\]

to obtain

\[
S = \begin{bmatrix} 20.6435 & 1.1922 & 4.6243 \\ 1.1922 & 4.5398 & 1.4142 \\ 4.6243 & 1.4142 & 6.4999 \end{bmatrix}
\]

Take \( \sigma = 1, r_z = 20, r_o = 150 \) and solve the algebraic Riccati equation

\[
P(A_z(p_o) + \sigma I)^T + (A_z(p_o) + \sigma I)P \\
- r_o PC_z^T C_z P + \frac{r_z}{2} B_z B_z^T = 0
\]

we obtain

\[
P = \begin{bmatrix} 0.1183 & 0.1265 & -0.0612 \\ 0.1265 & 2.4446 & 0.1183 \\ -0.0612 & 0.1183 & 0.0745 \end{bmatrix}
\]

Since

\[
-2\sigma P^{-1} - r_o C_z^T C_z + \frac{1}{r_z} SB_z B_z^T S
\]

\[
= \begin{bmatrix} -193.4835 & 4.5865 & -192.5306 \\ 4.5865 & -0.2834 & 5.9508 \\ -192.5306 & 5.9508 & -220.6638 \end{bmatrix} < 0
\]

by Theorem 6, the solution to the robust tracking exists and is given by

\[
K_z = -B_z^T S = \begin{bmatrix} -0.5961 \\ 2.2699 \\ 0.7071 \end{bmatrix} \\
L = -r_o PC_z^T = \begin{bmatrix} 8.5695 \\ 36.7185 \\ 2.0000 \end{bmatrix}
\]

The simulation results are shown in Figure 7.

If we want a faster tracking by placing all the poles of the system which are on the left of \( -\gamma \), then we can use the following theorem.

Theorem 7: Assume that the matching condition is satisfied and there exists a nominal value \( p_o \in P \) such that \((A_z(p_o), B_z)\) is controllable and \((A_z(p_o), C_z)\) is observable. If there exist design parameters \( \sigma > 0, r_z > 0 \) and \( r_o > 0 \) such that the solutions to the algebraic Riccati equations

\[
(A_z^T(p_o) + \gamma I)S + S(A_z(p_o) + \gamma I) - \frac{1}{2} SB_z B_z^T S + G + I = 0
\]

\[
P(A_z(p_o) + (\gamma + \sigma)I)^T + (A_z(p_o) + (\gamma + \sigma)I)P \\
- r_o PC_z^T C_z P + \frac{r_z}{2} B_z B_z^T = 0
\]

satisfy

\[
-2(\gamma + \sigma) P^{-1} - r_o C_z^T C_z + \frac{1}{r_z} SB_z B_z^T S < 0
\]

then the solution to the robust tracking problem based on observer exists and is given by

\[
K_z = -B_z^T S \\
L = -r_o PC_z^T
\]

Furthermore, all the poles of the system are on the left of \( -\gamma \).

Proof: The proof is similar to that of Theorem 3. □
Example 6: Consider the system discussed in Example 5. We would like to design an observer-based feedback control to ensure that the output $y$ is tracking a reference signal $y_r = 1$ with $y = 1$ for all $p \in [-2, 2]$. The augmented system is the same as that of Example 5. We solve the algebraic Riccati equation

$$(A_2(p_o) + \gamma I)^T S + S(A_2(p_o) + \gamma I) - \frac{1}{2} SB_2 B_2^T S + G + I = 0$$

to obtain

$$S = \begin{bmatrix} 188.5285 & 32.4346 & 203.7107 \\ 32.4346 & 12.7960 & 37.7381 \\ 203.7107 & 37.7381 & 355.5406 \end{bmatrix}.$$  

Take $\sigma = 0.01$, $r_c = 8000$, $r_o = 1500$ and solve the algebraic Riccati equation

$$P(A_2(p_o) + (\sigma + \gamma)I)^T + (A_2(p_o) + (\sigma + \gamma)I)P - r_o PC_2^T C_2 P + \frac{r_c}{2} B_2 B_2^T = 0$$

we obtain

$$P = \begin{bmatrix} 5.0679 & -3.9717 & -5.0288 \\ -3.9717 & 120.5163 & 5.6117 \\ -5.0288 & 5.6117 & 5.0377 \end{bmatrix}.$$  

Since

$$-2(\gamma + \sigma)P^{-1} - r_o C_2^T C_2 + \frac{1}{r_c} SB_2 B_2^T S$$

$$= \begin{bmatrix} -1581.5696 & 1.2174 & -1582.7023 \\ 1.2174 & -0.0138 & 1.262 \\ -1582.7023 & 1.262 & -1584.2706 \end{bmatrix} < 0$$

by Theorem 7, the solution to the robust tracking exists and is given by

$$K_z = -B_2^T S = \begin{bmatrix} -16.2173 & 6.3980 & 18.8690 \end{bmatrix}$$

$$L = -r_o PC_2^T = \begin{bmatrix} 58.6562 & 2459.9342 & 13.3341 \end{bmatrix}^T.$$  

The simulation results are shown in Figure 8.

For the simulation, we know that the system response is faster than that in Figure 7.

5. Conclusion

In this article, we investigated the robust tracking problem. The signal to be tracked is assumed to be a polynomial function of time. New state variables were introduced to represent errors and their integrals. Two types of control were considered: state feedback control and output feedback control. For state feedback control, a state feedback matrix is to be designed. For output feedback control, both state
feedback matrix and observer are to be designed. The design was based on an optimal control approach and by solving algebraic Riccati equations. The closed-loop systems were proved to be robustly stable and track the desired signals. We considered both matched uncertainties and unmatched uncertainties for state feedback, but only matched uncertainties for output feedback.

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References


