Robust Disturbance Attenuation with Stability for Linear Systems with Norm-Bounded Nonlinear Uncertainties

Le Yi Wang and Wei Zhan

Abstract—In this paper, the problem of robust stabilization and robust disturbance attenuation is investigated for systems with linear nominal parts and norm-bounded nonlinear uncertainties on both states and control inputs. It is shown that the type of nonlinear uncertainty sets considered in this paper has an equivalent representation by linear uncertainty sets. Based on this key result and some standard Riccati inequality approaches for robust control of linear uncertain systems, a constructive design procedure is developed.

I. INTRODUCTION

This paper is concerned with the problem of robust stabilization and robust disturbance attenuation for systems with linear nominal parts and norm-bounded nonlinear uncertainties.

Significant progress in robust control theory for linear systems, especially the development of \(H^\infty\) theory, during the past decade has spurred great interest in extending newly established results to nonlinear systems. In particular, the numerical accessibility and efficiency of the Riccati equation approach for \(H^\infty\) design of linear time-invariant systems has motivated the search of its counterpart in nonlinear systems. This search has lead to some recent publications on nonlinear \(H^\infty\) control [1], [8], [13]. While the resulting Hamilton–Jacobi–Isaac (HJI) inequalities resemble Riccati inequalities in their symbolic representations, solutions to the HJI inequalities and construction of controllers are usually extremely difficult to obtain. Furthermore, at the current stage of development, the HJI approach of \(H^\infty\) control concentrates on stabilization and disturbance attenuation of nominal plants. The problem of robust stabilization and robust performance remains largely unresolved.

This paper presents new results on control synthesis for robust stabilization and robust disturbance attenuation for linear systems with norm-bounded nonlinear uncertainties, continuing the work of [14]–[16]. The class of plants considered in this paper consists of systems in state-space form with linear nominal parts and norm-bounded nonlinear uncertainties on both states and control inputs. Robust stabilization and robust disturbance attenuation of such systems are investigated using the Hamiltonian approach. We demonstrate that the type of norm-bounded nonlinear uncertainties considered in this paper coincides with a set of linear uncertainties (Lemma 2). As a result, the corresponding HJI inequality can be reduced to a certain Riccati inequality. Then, certain standard \(H^\infty\) design techniques for linear systems can be applied (e.g., [10] and [12]). Consequently, control synthesis can be achieved by solving the corresponding Riccati inequalities (Theorem 1) which are readily solvable, at least numerically.

The main results of this paper are related to the following existing results on linear and nonlinear \(H^\infty\) control. References [10], [12], and [17] developed constructive design procedures for robust quadratic stabilization of linear systems with time-varying uncertainties. References [4] and [5] studied nominal disturbance attenuation problems.

II. PROBLEM FORMULATION

Consider a system in state-space form

\[
\begin{align*}
\dot{x} &= Ax + B_1 u + B_2 d + \Delta(x, u) \\
y &= Cx
\end{align*}
\]

where \(x \in \mathbb{R}^n, u \in \mathbb{R}^m, d \in \mathbb{R}^k, y \in \mathbb{R}^l\) are state, control input, disturbance, and output, respectively. \(\Delta(x, u)\) is a nonlinear term of uncertainty. The matrices \(A, B_1, B_2, C, \Delta\) have compatible dimensions. For a vector \(v \in \mathbb{R}^p, v^T\) is its transpose, and \(\|v\|\) its Euclidean norm. For a matrix \(M \in \mathbb{R}^{p \times q}, \sigma(M)\) will denote its singular value. The uncertainty term \(\Delta(x, u)\) is assumed to be norm bounded: for some \(\varepsilon_1 \geq 0, \varepsilon_2 \geq 0\)

\[
\|\Delta(x, u)\| \leq \varepsilon_1 \|x\| + \varepsilon_2 \|u\|,
\]

for all \(x \in \mathbb{R}^n, u \in \mathbb{R}^m\) but otherwise arbitrary. Denote the corresponding uncertainty set by

\[
\Omega(x, u) = \{ \Delta(x, u) : \|\Delta(x, u)\| \leq \varepsilon_1 \|x\| + \varepsilon_2 \|u\|\}.
\]

Definition (Robust Stability and Robust Disturbance Attenuation):

1) A state feedback \(u = -Kx, K \in \mathbb{R}^{m \times n}\) is said to achieve robust global asymptotic stability if for \(d = 0\) and any \(\Delta(x, u) \in \Omega(x, u)\), the closed-loop system

\[
\dot{x} = (A - B_1 K)x + \Delta(x, -Kx)
\]

is globally asymptotically stable in the Lyapunov sense.

2) A state feedback \(u = -Kx, K \in \mathbb{R}^{m \times n}\) is said to achieve robust disturbance attenuation if under zero initial condition \(x(0) = 0\), there exists \(0 \leq \gamma < \infty\) for which the performance bound

\[
\int_0^T (y^T W_y y + u^T W_u u) dt \leq \gamma^2 \int_0^T (d^T W_d d) dt,
\]

for all \(d \in L^2\) (the Lebesgue space of square integrable functions) and all \((x, u) \in \Omega(x, u)\), where the weighting matrices are assumed to be symmetric and satisfy

\[
\begin{align*}
W_y &\geq 0 \\
W_u &> 0 \\
W_d &> 0
\end{align*}
\]

Problem (P): Design \(K \in \mathbb{R}^{m \times n}\) such that the state feedback \(u = -Kx\) achieves robust global asymptotic stability and robust disturbance attenuation.

The main approach employed here is the standard HJI method. Hence, we define a quadratic energy function

\[
E(x) = x^T P x
\]

where \(P > 0\) is to be determined. Define the Hamiltonian function

\[
H[u, d, \Delta(x, u)] = y^T W_y y - \gamma^2 d^T W_d d + u^T W_u u + \frac{dE}{dt}
\]
where \( \frac{dE}{dt} \) is the derivative of \( E \) along the trajectory of the closed-loop system. It is well known that a sufficient condition for achieving robust disturbance attenuation (2) is that
\[
H[u, d, \Delta(x, u)] < 0, \quad \text{for all} \quad d \in L^2, \Delta(x, u) \in \Omega(x, u).
\]
(4)

Also, under (4), \( E(x) \) is a strict radially unbounded Lyapunov function of the closed-loop system, and hence robust stability is guaranteed. In this paper we will establish conditions under which
\[
\inf_u \sup_{\Delta(x, u) \in \Omega(x, u)} H[u, d, \Delta(x, u)] < 0.
\]
(5)

III. MAIN RESULTS

The key technical contribution of this paper is Lemma 2 which establishes a representation of the nonlinear uncertainty set \( \Omega(x, u) \) by a linear uncertainty set. This key result is then combined with the standard HJI approach to derive the main result (Theorem 1) which provides a constructive robust design procedure for solving Problem (P).

A. Basic Lemmas

Lemma 1: For \( m \leq n \), suppose \( v \in \mathbb{R}^n \) with \( \|v\| = 1 \), and \( u \in \mathbb{R}^m \) with \( \|u\| = 1 \). Then there exists \( M \in \mathbb{R}^{m \times n} \) with \( \sigma(M) \leq 1 \) such that
\[
v = Mu.
\]

Proof: Since \( v \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) have unit norm, we can construct orthonormal basis via, say, the classical Gram–Schmidt algorithm
\[
V = [v_1, v_2, \ldots, v_m] \quad \text{for} \quad \mathbb{R}^n
\]
and
\[
U = [u_1, u_2, \ldots, u_m] \quad \text{for} \quad \mathbb{R}^m.
\]

Obviously, \( V_m := [v_1, v_2, \ldots, v_m] \) satisfies \( V_m^T V_m = I \). It follows that \( M := V_m U^T \) satisfies \( M^T M = I \) which implies \( \sigma(M) \leq 1 \). Since \( MU = V_m u \), we have \( v = Mu \) as required.

Let \( \Omega_1(x, u) \) be the following linear uncertainty set
\[
\Omega_1(x, u) := \{ \xi_1 M_1 x + \xi_2 M_2 u : M_1 \in \mathbb{R}^{n \times n}, M_2 \in \mathbb{R}^{n \times m}, \sigma(M_1) \leq 1, \sigma(M_2) \leq 1 \}.
\]

Lemma 2:
\[
\Omega(x, u) = \Omega_1(x, u).
\]

Proof: It is obvious that
\[
\Omega(x, u) \subseteq \Omega_1(x, u).
\]
Moreover, for any \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \)
\[
\Omega(x, u) \subseteq \Phi = \{ v \in \mathbb{R}^n : \|v\| \leq \epsilon_1 \|x\| + \epsilon_2 \|u\| \}.
\]
(7)

Suppose \( v \in \Phi \). Then there exist \( a_1 \leq \epsilon_1, a_2 \leq \epsilon_2 \) such that
\[
\|v\| = a_1 \|x\| + a_2 \|u\|.
\]
(8)

Decompose
\[
v = \alpha_1 v + \alpha_2 u
\]
where
\[
\alpha_1 = \frac{a_1 \|x\|}{\|v\|}, \quad \alpha_2 = \frac{a_2 \|u\|}{\|v\|}.
\]
(9)

Without loss of generality, assume \( x \neq 0, u \neq 0 \). Define
\[
\dot{v} = \frac{v}{\|v\|}, \quad \dot{x} = \frac{x}{\|x\|}, \quad \dot{u} = \frac{u}{\|u\|}.
\]

By Lemma 1, there exist \( M_1 \in \mathbb{R}^{n \times n} \) and \( M_2 \in \mathbb{R}^{n \times m} \) with \( \sigma(M_1) \leq 1, \sigma(M_2) \leq 1 \) such that
\[
\dot{v} = M_1 \dot{x}, \quad \dot{u} = M_2 \dot{u}
\]
which, together with (9), leads to
\[
\alpha_1 = a_1 M_1 x, \quad \alpha_2 = a_2 M_2 u.
\]

It follows that
\[
v = \alpha_1 v + \alpha_2 u = a_1 M_1 x + a_2 M_2 u \in \Omega_1(x, u).
\]

Since \( v \in \Phi \) is arbitrary, we conclude that
\[
\Phi \subseteq \Omega_1(x, u)
\]
which, along with (7), implies
\[
\Omega(x, u) \subseteq \Omega_1(x, u).
\]

B. Main Theorem

By Lemma 2, robust stabilization and robust disturbance attenuation of (1) can be achieved by employing design techniques for linear uncertain systems. This observation leads to Theorem 1, the main result of this paper. The main approach employed here is the standard method of Riccati inequalities which have been used extensively in linear \( H^\infty \) control for state-space systems (see, e.g., [10], [12], and [17]).

Theorem 1: If there exist positive numbers \( \xi_1 > 0, \xi_2 > 0 \) and a positive definite solution \( P > 0 \) to the Riccati inequality
\[
C^T W_1 C + PA + A^T P + \frac{1}{\gamma^2} PB_2 W_2^{-1} B_2 P
\]
\[
+ (\xi_1 \xi_2 + \xi_2 \xi_1) P^2 + \frac{\xi_2}{\xi_1} I
\]
\[
- PB_1 \left( W_u + \frac{\xi_2}{\xi_1} I \right)^{-1} B_1^T P < 0
\]
(11)
then the control law
\[
u = - \left( W_u + \frac{\xi_2}{\xi_1} I \right)^{-1} B_1^T P x
\]
(12)
achieves robust global asymptotic stability and robust disturbance attenuation
\[
\int_0^T (g^T W_g y + u^T W_u u) dt \leq \gamma^2 \int_0^T (d^T W_d d) dt,
\]
\[
\forall T \geq 0
\]
(13)
for all \( d \in L^2 \) and all \( \Delta(x, u) \in \Omega(x, u) \).

Remark: Since (11) is numerically solvable, Theorem 1 provides a design procedure for robust stabilization and robust disturbance attenuation of systems with linear nominal parts and norm-bounded nonlinear uncertainties.
Proof of Theorem 1: We will prove the theorem by showing that the control law (12) will guarantee that
\[ H[u, d, \Delta(x, u)] < 0 \]
for all \( u \in L^2 \), \( \Delta(x, u) \in \Omega(x, u) \).
Since \( W_d > 0 \), it can be decomposed into
\[ W_d = A_d^T A_d \]
where \( A_d \) is nonsingular. By definition
\[ H[u, d, \Delta(x, u)] = y^T W_y y - \gamma^2 d^T W_d d + u^T W_u u + dE \quad \text{for all } d \in L^2, \Delta(x, u) \in \Omega(x, u). \]
From the expression of \( E(x) \), we can easily derive, noting that \( y = C x \)
\[ H[u, d, \Delta(x, u)] = x^T C^T W_y C x + x^T (PA + A^T P)x \]
\[ - \gamma^2 d^T W_d d + x^T P B_2 d + d^T B_d^T P d \]
\[ + u^T W_u u + x^T P B_1 u + u^T B_1^T P x \]
\[ + x^T P \Delta(x, u) + x^T (x, u) P x. \]
It is easy to show that the worst case sup over \( \Delta(x, u) \) occurs when
\[ d = \frac{1}{\gamma^2} W_d^{-1} B_d^T P x. \]
It follows that
\[ H_1[u, \Delta(x, u)] = \sup_{d \in L^2} H[u, d, \Delta(x, u)] \]
\[ = F(x) + u^T W_u u + x^T P B_1 u \]
\[ + u^T B_1^T P x + x^T P \Delta(x, u) + x^T (x, u) P x. \]
where
\[ F(x) := x^T [C^T W_y C + PA + A^T P]x \]
\[ + \frac{1}{\gamma^2} P B_2 W_d^{-1} B_d^T P x. \]
By Lemma 2
\[ \sup_{\Delta(x, u) \in \Omega(x, u)} H_1[u, \Delta(x, u)] = \sup_{\Delta(x, u) \in \Omega(x, u)} H_1[u, \Delta(x, u)]. \]
Hence, we only need to consider
\[ H_1[u, \Delta(x, u)] = F(x) + u^T W_u u + x^T P B_1 u \]
\[ + u^T B_1^T P x + x^T P \Delta(x, u) + x^T (x, u) P x. \]
Now, it is trivial to show [10] that for any \( \varepsilon_a > 0 \) and \( \varepsilon_b > 0 \)
\[ x^T P \Delta(x, u) + x^T (x, u) P x \leq \varepsilon_a \Delta^T P^2 x + \varepsilon_b x^T x \]
and
\[ x^T P \varepsilon_2 M_2 u + (x^T \varepsilon_2 M_2 u)^T P x \leq \varepsilon_a x^T P^2 x + \varepsilon_b x^T u. \]
Consequently
\[ \sup_{\Delta(x, u) \in \Omega(x, u)} H_1[u, \Delta(x, u)] \leq \inf_{u} \sup_{\Delta(x, u) \in \Omega(x, u)} \]
\[ F(x) + (\varepsilon_1 \varepsilon_a + \varepsilon_2 \varepsilon_a) x^T P^2 x + \varepsilon_1 \varepsilon_a x^T x + u^T \left( W_u + \varepsilon_2 \varepsilon_b \right) u \]
\[ + x^T P B_1 u + u^T B_1^T P x. \]
The optimal control law which minimizes the right-hand side is given by
\[ u = - \left( W_u + \varepsilon_2 \varepsilon_b \right)^{-1} B_1^T P x. \]
and
\[ \inf_{u} \sup_{\Delta(x, u) \in \Omega(x, u)} \]
\[ F(x) + (\varepsilon_1 \varepsilon_a + \varepsilon_2 \varepsilon_a) x^T P^2 x + \varepsilon_1 \varepsilon_a x^T x + u^T \left( W_u + \varepsilon_2 \varepsilon_b \right) u \]
\[ + x^T P B_1 u + u^T B_1^T P x. \]
Consequently, if there exists a positive definite solution \( P > 0 \) to the Riccati inequality
\[ C^T W_y C + PA + A^T P + \frac{1}{\gamma^2} P B_2 W_d^{-1} B_d^T P \]
\[ + (\varepsilon_1 \varepsilon_a + \varepsilon_2 \varepsilon_b) P^2 \]
\[ + \varepsilon_1 \varepsilon_a + \varepsilon_2 \varepsilon_b \]
\[ P < 0 \]
then we have
\[ H[u, d, \Delta(x, u)] < 0 \]
for all \( u \in L^2 \) and \( \Delta(x, u) \in \Omega(x, u) \).

IV. CONCLUDING REMARKS

It is noted that while the underlying systems are nonlinear, the control law employed here is linear. A question arises: Is such a design overly conservative? The authors would like to conjecture that for the class of systems considered in this paper, robust stabilization and disturbance attenuation achievable by nonlinear static state feedback can be achieved by linear static state feedback. This conjecture is being investigated.

REFERENCES

[16] W. Zhan, L. Y. Wang, J. Liu, and J. Sun, “\( H^\infty \) control for a class of systems with sector bound nonlinearity,” in Proc. NOLCOS 95.