Continuity of Optimal Robustness and Robust Stabilization in Slowly Varying Systems*

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Abstract—Continuity properties of the optimal design of robust stabilization in the gap metric are investigated. In general the optimal design in the gap metric lacks continuity properties required for certain \( H^\infty \) adaptation schemes for slowly varying systems. On the other hand, the \( \delta \)-suboptimal central design is shown to be Lipschitz continuous. Applications of the continuity properties and frozen-time stability analysis lead to a suboptimal design for robust stabilization in slowly time-varying plants which are represented by local normalized coprime factorizations.

1. INTRODUCTION

This paper is concerned with the problem of robust stabilization in slowly time-varying systems, continuing the work started in Wang and Zames (1989). The development is carried out in the local–global double algebra framework (Zames and Wang, 1991) using frozen-time analysis and local \( H^\infty \) design. The main results of the paper were reported in Wang and Joh (1992).

The frozen-time approach was introduced into the control field in the 1960s (Freedman and Zames, 1968; Desoer, 1970), initially for stability analysis in slowly varying systems on a case–by–case basis. In the frozen-time approach, a time-varying system is modeled as a sequence of shift-invariant systems acting locally at frozen time. The design of feedback controllers is then reduced to the construction of feedback mappings from frozen-time models of the plant to those of the controller, enabling, at least for slowly varying systems, the analysis and synthesis of time-varying systems to be carried out in the domain of time-invariant systems for which a large inventory of designing tools is available. The frozen-time design obeys the design causality principle, namely only past and present information about the plant can be employed at the present stage of control synthesis. Since the design causality principle is an essential constraint for adaptation, the frozen-time approach is of importance in developing a comprehensive theory of adaptation. The potential impact on adaptation theory has spurred renewal interest in the approach during the past decade. Some major progress has been made in extending the approach to stabilization (Kamen et al., 1989), performance optimization (Wang and Zames, 1989; Dahleh and Dahleh, 1990, 1991; Zames and Wang, 1991) and adaptation (Voulgaris et al., 1992) for slowly time-varying systems.

A challenging problem in employing the frozen-time approach to achieve robust stabilization is to design a feedback controller which permits uniform frozen-time robust stability and is itself slowly varying. One possible way of tackling this problem is to maintain Lipschitz continuity of the feedback mapping from frozen-time models of the plant to those of the controller, enabling, at least for slowly varying systems, the analysis and synthesis of time-varying systems to be carried out in the domain of time-invariant systems for which a large inventory of designing tools is available. The frozen-time design obeys the design causality principle, namely only past and present information about the plant can be employed at the present stage of control synthesis. Since the design causality principle is an essential constraint for adaptation, the frozen-time approach is of importance in developing a comprehensive theory of adaptation. The potential impact on adaptation theory has spurred renewal interest in the approach during the past decade. Some major progress has been made in extending the approach to stabilization (Kamen et al., 1989), performance optimization (Wang and Zames, 1989; Dahleh and Dahleh, 1990, 1991; Zames and Wang, 1991) and adaptation (Voulgaris et al., 1992) for slowly time-varying systems.

A challenging problem in employing the frozen-time approach to achieve robust stabilization is to design a feedback controller which permits uniform frozen-time robust stability and is itself slowly varying. One possible way of tackling this problem is to maintain Lipschitz continuity of the feedback mapping from frozen-time models of the plant to those of the controller, which will then produce a slowly varying controller whenever the plant varies slowly. Continuity properties of feedback design are, in general, of importance in studies of wellposedness and robustness of feedback control, in understanding interactions between modeling and feedback action, and in the development of a comprehensive theory of adaptation. In this paper (Section 3), the problem of Lipschitz continuity of feedback design achieving optimal or suboptimal robustness in the gap metric is studied (Zames and El-Sakkary, 1980; Georgiou and Smith, 1990).
The particular type of Lipschitz continuity investigation in this paper arises naturally in certain schemes of $H^\infty$ adaptation (Zames and Wang, 1991). It is demonstrated in this paper, by using an example from Georgiou and Smith (1992), that the optimal robustness design in the gap metric is not Lipschitz continuous. On the other hand, it is shown that the $\delta$-suboptimal central solutions to robust stabilization in the gap metric is Lipschitz continuous, and can be employed to derive a frozen-time adaptive design for stabilization of slowly varying plants which are represented by local normalized coprime factorizations. The main results of Section 3 are Theorems 1 and 2 which provide explicit bounds on the Lipschitz constants of the feedback mapping defined by the suboptimal design in the gap metric, revealing an intrinsic trade-off between optimality and continuity.

Stability analysis for slowly varying systems is investigated in Section 4. Using the local-global double algebra framework introduced in Wang and Zames (1989) and Zames and Wang (1991), sufficient conditions are obtained (Lemma 3) for BIBO stability when systems are frozen-time exponentially stable and variation rates of both plants and controllers are small. Explicit bounds on variation rates for stability are established. Applications of Theorems 1 and 2 and Lemma 3 lead to a design procedure of robust stabilization for slowly varying plants (Section 5). The design procedure is shown to achieve BIBO stability in Theorem 3 and robust stability in Theorem 4, for plants which are represented by local normalized coprime factorizations with sufficiently small variation rates.

1.1. Related work

The frozen-time approach for stability analysis was initiated in Freedman and Zames (1968) and Desoer (1970). Substantial progress has since been made to extend the approach to stabilization, performance optimization and adaptation. Stabilization for slowly varying state-space systems was investigated in Kamen et al. (1989) in which continuity properties of the underlying feedback were established and applied to derive a design procedure. A double algebra framework was introduced in Wang and Zames (1989) and Zames and Wang (1991) for studies of stabilization and optimization in slowly time-varying systems and of slow $H^\infty$ adaptation. Within the $l^1$ optimization framework, stability analysis and synthesis in slowly varying systems were obtained in Dahleh and Dahleh (1990, 1991). Recently, the results of Dahleh and Dahleh (1990, 1991) were combined with certain identification algorithms in Voulgaris et al. (1992) to obtain a novel robust adaptive design procedure.

Continuity properties of $H^\infty$ interpolation were studied in Smith (1990), Wang and Zames (1990) and Wang (1991). The gap metric was introduced into control theory by Zames and El-Sakkary (1980) to study uncertainty and robust stability of feedback systems. In Georgiou and Smith (1990), the gap metric was employed to develop a theory of robust stabilization for linear time-invariant systems. Continuity properties of the optimal robustness in the gap metric were discussed in Georgiou and Smith (1992).

2. PRELIMINARIES

The notation employed in this paper is fairly standard. $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the reals, integers and complex numbers. $\mathbb{C}^{n \times m}$ denotes the space of $n \times m$ matrices of elements in $\mathbb{C}$. For $K \in \mathbb{C}^{n \times m}$, $K^*$ is its conjugate transpose, $|K|$ its largest singular value and $|K|_2$ its Euclidean norm.

2.1. Systems

Input-output spaces will be $l^\infty$, the Banach space of $\mathbb{C}$-valued functions $u$ for which

$$\|u\|_\infty := \sup_{t \in \mathbb{Z}} |u(t)|_2 < \infty.$$  

Stable systems will belong to the Banach space $\mathcal{B}$ of bounded causal linear operators $K : l^\infty \to l^\infty$ which have convolution sum representations

$$(Ku)(t) = \sum_{r=-\infty}^{t} k(t, r)u(r), \quad t \in \mathbb{Z}$$  

where the kernel $k : \mathbb{Z}^2 \to \mathbb{C}^{n \times n}$ is assumed, for each $t \in \mathbb{Z}$, to satisfy $k(t, \cdot) \in l^1$, i.e.

$$\|K\|_{\mathcal{B}} := \sup_{t \in \mathbb{Z}} \|k(t, \cdot)\|_1$$

$$= \sup_{t \in \mathbb{Z}} \sum_{r=-\infty}^{\infty} |k(t, r)| < \infty,$$

and $k(t, \tau) = 0$ whenever $t < \tau$. The norm on $\mathcal{B}$ will be $\|\cdot\|_{\mathcal{B}}$.

Shift-invariant systems $K$ in $\mathcal{B}$ have well-defined frequency responses $K$ in $H^\infty$, the Hardy space of bounded analytic functions on the unit disk. In general, for $p = 2$ or $\infty$, and $\sigma \geq 1$, $(H^p)^{n \times m}$ is the Hardy space of $(n \times m$ matrix)-valued analytic functions on the open disk of radius $\sigma$, which will be viewed as a subspace of the usual Lebesgue space $(L^p)^{n \times m}$ of functions on the circle of radius $\sigma$. The norm
in $\mathcal{L}^p_{\nu} \otimes \mathcal{L}^q$ is

$$
\|K\|_{\mathcal{L}^p} = \left\{ \begin{array}{ll}
\sup_{\theta \in [-\pi, \pi]} |K(\nu e^{j\theta})|, & p = \infty, \\
\left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |K(\nu e^{j\theta})|^p d\theta \right)^{1/p}, & p = 2.
\end{array} \right.
$$ (2)

The superscript $n \times m$ will usually be suppressed from the notation if it is apparent from the context. Also, the subscript $\sigma$ will be omitted when $\sigma = 1$, as in $H^n$, and $\|K\|_2 = \|K\|_{\mathcal{L}^2}$.

Unstable systems will belong to $B\sigma$, the algebra of discrete-time $n$ input $n$ output linear causal systems. $B\sigma$ denotes the ring of systems in $\mathcal{H}$ which have inverses in $\mathcal{H}$.

2.2. Coprime factorization

A pair $(N, D) \in B \times \mathbb{R}$ is right coprime in $B$ if for some $X, K \in \mathbb{R}$ the Bezout equation

$$
XN + YD = I,
$$ (3)

is satisfied (I is the identity operator). The pair $(X, Y) \in \mathbb{R} \times \mathbb{R}$ is then called left coprime. A system $G \in B\sigma$ has a right (or left) factorization representation in $B\sigma$ if $G = ND^{-1}$ (or $G = \tilde{D}^{-1}N$), where $(N, D)$ (or $(\tilde{N}, \tilde{D})$) $\in B \times B\sigma$. The factorization $ND^{-1}$ (or $\tilde{D}^{-1}N$) is right (or left) coprime if the corresponding pair $(N, D)$ (or $(\tilde{N}, \tilde{D})$) is right (or left) coprime.

The interconnection of a feedback controller $F$ and a plant $P$ in $B\sigma$ is well posed if

$$
K = [K_0] := \begin{pmatrix} (I + PF)^{-1} P(I + FP)^{-1} \\ (I + PF)^{-1} (I + FP)^{-1} \end{pmatrix}
$$ (4)

is in $\mathcal{B}\sigma$. The pair $(F, P)$ is then said to be well posed. If, in addition, $K \in \mathbb{R}$, then $K$ is said to be stable and $F$ a stabilizing controller for $P$.

The famous Youla parametrization (Youla et al. (1976); Desoer et al. (1980)), which is restated in Proposition 1 without proof, provides a complete characterization of all stabilizing feedback controllers for a given plant.

**Proposition 1.** (Youla et al. (1976) and Desoer et al. (1980)). Suppose $P \in B\sigma$ admits left and right coprime factorization representations in $B\sigma$, $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ with $X_0, Y_0, \tilde{X}_0, \tilde{Y}_0 \in B$

$$
\tilde{X}_0N + \tilde{Y}_0D = I.
$$ (5)

Then, all stabilizing feedback controllers for $P$ are parametrized by

$$
F = (X_0 + DQ)(Y_0 - NQ)^{-1},
$$ (6)

where $Q \in \mathbb{B}$ is a free parameter subject to $Y_0 - NQ \in \mathbb{B}$.

2.3. Lipschitz continuity of feedback design

This section concentrates on the case of shift-invariant systems. Let $\mathcal{H}^s$ denote the algebra of linear time-invariant discrete-time systems with frequency responses in $(\mathcal{H})^s$. $\mathcal{H}^s$ is contained in the larger algebra $\mathcal{H}^s$ of all $n$-input--$n$-output linear causal time-invariant discrete-time systems. $\mathcal{H}^s$ will denote the ring of systems in $\mathcal{H}^s$ which are invertible in $\mathcal{H}^s$. For a system $G \in \mathcal{H}^s$, $G$ will denote its frequency response. Shift-invariant systems in $B\sigma$ and $B\sigma$ belong to $\mathcal{H}^s$ and $\mathcal{H}^s$, respectively. Discussions in the previous section carry through here with $B\sigma$ and $B\sigma$ replaced by $\mathcal{H}^s$ and $\mathcal{H}^s$, respectively.

For $P \in \mathcal{H}^s$ satisfying the assumptions of Proposition 1, $S(P)$ will denote the set of all stabilizing controllers for $P$ given by (6). Let $IA$ be the set of all systems in $\mathcal{H}^s$ admitting left and right coprime factorization representations in $\mathcal{H}^s$. It is apparent from Proposition 1 that if $P \in IA$ then $S(P) \subseteq IA$. In other words, $IA$ is invariant under the feedback mapping from plants to stabilizing controllers.

As evidenced by Proposition 1, stabilizing controllers for a given plant in $IA$ are not unique. To study continuity properties of stabilizing feedback mappings, it is necessary to specify a unique assignment of stabilizing controllers to corresponding plants. Let $B:IA \rightarrow IA$ be a rule of assigning a stabilizing controller $F = XY^{-1} \in S(P)$ to $P = D^{-1}N \in IA$. The rule $B$ is said to be ‘Lipschitz continuous’ from $\mathcal{H}^s$ to $\mathcal{H}^s$ over a subset $\Omega \subseteq IA$ if for all $P_1, P_2 \in \Omega$

$$
\|B(P_1) - B(P_2)\|_2 := \left\| \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} - \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \right\|_2 
\leq \zeta \|P_1 - P_2\|_2 
:= \zeta \left\| \begin{pmatrix} N_1 \\ D_1 \end{pmatrix} - \begin{pmatrix} N_2 \\ D_2 \end{pmatrix} \right\|_2,
$$ (7)

where $\zeta$ is a constant independent of $P_1, P_2 \in \Omega$. This type of Lipschitz continuity from $\mathcal{H}^s$ to $\mathcal{H}^s$ arises naturally in certain $\mathcal{H}^s$ adaptation schemes for stabilization of slowly time-varying systems (Zames and Wang, 1991), and will be elaborated on in Section 4.

3. Lipschitz continuity of optimal robustness in the gap metric

3.1. Gap metric

Coprime representations† of $K \in IA$, $K = ND^{-1}$ are viewed as elements in $\mathcal{H}^s \times \mathcal{H}^s$ which is

† The version of left coprime factorization can be similarly defined and is omitted here for simplicity.
equipped with the norms $\| \cdot \|_1$ and $\| \cdot \|_2$ defined in (2), and the gap metric $\mathcal{G}$ (Zames and El-Sakkary, 1980; Georgiou and Smith, 1990). When $K_i = N_i D_i^{-1}$, $i = 1, 2$ are normalized coprime factorizations, i.e.

$$N_i^* N_i + D_i^* D_i = I,$$

the gap is

$$\mathcal{G} \left[ \begin{bmatrix} N_1 \\ D_1 \end{bmatrix}, \begin{bmatrix} N_2 \\ D_2 \end{bmatrix} \right] = \max \{ \mathcal{G}_1, \mathcal{G}_2 \},$$

where

$$\mathcal{G}_i = \inf_{Q \in \mathcal{H}^\infty} \left\| \begin{bmatrix} N_1 \\ D_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ D_2 \end{bmatrix} Q \right\|_\infty,$$

and

$$\mathcal{G}_2 = \inf_{Q \in \mathcal{H}^\infty} \left\| \begin{bmatrix} N_2 \\ D_2 \end{bmatrix} - \begin{bmatrix} N_1 \\ D_1 \end{bmatrix} Q \right\|_\infty.$$

Note that the distance between $K_1$ and $K_2$ in $\mathcal{IA}$ is dependent on representations when measured by $\| \cdot \|_1$ or $\| \cdot \|_2$. As a result, the error between coprime factorizations does not result in a metric on systems. On the other hand, the distance is representation invariant when measured by the gap, leading to the gap metric on systems in $\mathcal{H}^\infty$.

3.2. Optimal robustness in the gap metric

One possible way of arriving at a feedback rule is to assign to $P \in \mathcal{IA}$ the controller which achieves optimal robustness in the gap metric. When $P = ND^{-1} = D^{-1} N$ are normalized coprime factorization representations, this rule selects the optimal controller from

$$\inf_{P = XY^{-1} \in \mathcal{S}(P)} \left\| X \right\|_\infty.$$

Such an assignment is unique when $P$ is finite-dimensional and has the appealing property of providing an optimal robust stability margin in the gap metric (Georgiou and Smith, 1990).

In general, however, the feedback rule defined by optimal robustness in the gap metric is not Lipschitz continuous from $\mathcal{H}^\infty$ to $\mathcal{H}^2$, as shown by the following example. The example was constructed in continuous-time by Georgiou and Smith (1992) to demonstrate pointwise discontinuity of optimal controllers in the gap metric. It should be pointed out that in Example 1, when $\varepsilon_i = 0$ the Hankel operators associated with the plants have repeated top singular values, which characterize continuity properties of optimal interpolants as demonstrated in Peller (1991).

Example 1. Consider the discrete-time systems

$$P_i = \frac{1 - z + \varepsilon_i}{1 + z}, \quad i = 1, 2,$$

where $\varepsilon_i > 0$ are small numbers to be specified later. It is easy to verify that $P_i = N_i D_i^{-1}$, with

$$N_i = \frac{1 - z + \varepsilon_i}{1 + z}, \quad D_i = \frac{1 - z}{1 + z},$$

where $b_i = \sqrt{1 + \varepsilon_i}$, $a_i = \sqrt{2} b_i - 1$, is a normalized coprime representation of $P_i$. Moreover,

$$X_0^0 = \frac{1 - z}{1 + z}, \quad Y_0^0 = 1 \in \mathcal{H}^\infty,$$

solve the Bezout equation

$$N_i X_i^0 + D_i Y_i^0 = 1.$$

Then, the optimal controller in the gap metric is

$$\left[ \begin{array}{c} X_i \\ Y_i \end{array} \right] = \left[ \begin{array}{c} X_0^0 \\ Y_0^0 \end{array} \right] - \left[ \begin{array}{c} D_i \\ -N_i \end{array} \right] Q,$$

where $Q \in \mathcal{H}^\infty$ is the optimal parameter from the two-block $\mathcal{H}^\infty$ optimization

$$\inf_{Q \in \mathcal{H}^\infty} \left\| \begin{bmatrix} X_0^0 \\ Y_0^0 \end{bmatrix} - \left[ \begin{array}{c} D_i \\ -N_i \end{array} \right] Q \right\|_\infty.$$  (8)

The two-block $\mathcal{H}^\infty$ optimization problem can be reduced to the Nehari distance problem

$$\inf_{Q \in \mathcal{H}^\infty} \left\| V_i - Q \right\|_\infty = \left[ \frac{1}{1 + \left( \inf_{Q \in \mathcal{H}^\infty} \| V_i - O \|_\infty \right)^2 } \right]^{1/2} = (1 + \| \Gamma_{\nu_i} \|^2)^{1/2},$$

where $V_i = D_i^* X_i^0 - N_i^* Y_i^0$, and $\Gamma_{\nu_i}$ is the Hankel operator with symbol $V_i$.

Define the ratio $\zeta$ as

$$\zeta = \frac{\left\| \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} - \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} \right\|_\infty}{\left\| \begin{bmatrix} N_2 \\ D_2 \end{bmatrix} - \begin{bmatrix} N_1 \\ D_1 \end{bmatrix} \right\|_\infty}.$$
Choose, in particular,
\[ \varepsilon_1 = e^{-n/2}, \quad \varepsilon_2 = e^{-(n+1)/2}, \]
which approach zero as \( n \to \infty \).

It can be shown that when \( n \to \infty \), the ratio \( \zeta \to \infty \), by reducing the feedback design to a case of \( H^n \) to \( H^2 \) discontinuity (Wang and Zames, 1990).

Figure 1 demonstrates the ratio \( \zeta \) as a function of \( n \) when the Nehari distance problem is solved within a \( \delta \)-suboptimality of \( \delta = 10^{-8} \).

It is apparent that when \( n \) becomes large or, equivalently, \( \varepsilon \) becomes small, the feedback mapping defined by the optimal design in the gap metric tends to be very sensitive to perturbations in the plant.

The discontinuity demonstrated in this example implies that in general the optimal robustness design in the gap metric may result in large changes in controllers even when perturbations in the plant representations are small. As a result, the corresponding feedback rule may produce undesirable fast-varying controllers when applied to \( H^- \) frozen-time design (further discussed in Section 4). To avoid this difficulty, we are seeking certain suboptimal feedback rules which possess the desired Lipschitz continuity properties.

3.3. Suboptimal design: plants with normalized coprime factorizations

We start, in this section, with plants modeled by normalized coprime factorization representations.

**Assumption 1.** Suppose \( P_i = N_iD_i^{-1} = \overline{D}_i^{-1}\overline{N}_i \), \( i = 1, 2 \) are normalized coprime factorization representations, \( N_i^*N_i + D_i^*D_i = I \) and \( \overline{N}_i\overline{N}_i^* + \overline{D}_i\overline{D}_i^* = I \),

and

\[
\overline{X}_i^0\overline{N}_i + \overline{Y}_i^0\overline{D}_i = I, \\
\overline{N}_i\overline{X}_i^0 + \overline{D}_i\overline{Y}_i^0 = I.
\]

for some \( X_i, Y_i, \overline{X}_i^0, \overline{Y}_i^0 \in H^n \).

By Proposition 1, all stabilizing feedback controllers for \( P_i \) can be parametrized as \( F = X_iY_i^{-1} \).

\[
X_i = X_i^0 + D_iQ_i \quad \text{and} \quad Y_i = Y_i^0 - N_iQ_i,
\]

where \( Q_i \in H^n \) is a free parameter. As shown in Georgiou and Smith (1990), the robustness optimization in the gap metric can be reduced to the two-block \( H^n \) optimization problem:

\[
\mu_i = \inf_{Q_i \in H^n} \left\| \begin{bmatrix} X_i^0 & -D_i \\ -N_i & Q_i \end{bmatrix} \right\|_\infty, \quad i = 1, 2. \quad (9)
\]

Furthermore, it is demonstrated in Glover and McFarlane (1989) and Georgiou and Smith (1990) that the optimization problem (9) can be reduced to the standard Nehari distance problem as follows:

\[
\mu_i = \inf_{Q_i \in H^n} \left\| V_i - Q_i \right\|_\infty \\
= \left( \inf_{Q_i \in H^n} \left\| V_i - Q_i \right\|_2^2 + 1 \right)^{1/2} \\
= (\|\Gamma_{V_i}\|^2 + 1)^{1/2},
\]

where \( V_i \) is the system with frequency response \( V_i = D_i^*X_i^0 - N_i^*Y_i^0 \) and \( \|\Gamma_{V_i}\| \) is the operator norm of the Hankel operator with symbol \( V_i \).

Let \( Q_i^0 \in H^n \) be the (unique) \( \delta \)-suboptimal central interpolant (see, e.g. Adamjan et al. (1978) and Wang and Zames (1990)) to the optimal interpolation problem

\[
\inf_{Q_i \in H^n} \left\| V_i - Q_i \right\|_\infty,
\]

i.e.,

\[
\left\| V_i - Q_i^0 \right\|_\infty \leq \|\Gamma_{V_i}\| + \delta.
\]

\( Q_i^0 \) can be constructed using, e.g. the AAK construction (Adamjan et al., 1978). \( Q_i^0 \) induces a (unique) suboptimal solution \( (X_i^0, Y_i^0) \) to the robustness optimization problem (9)

\[
\begin{bmatrix} X_i^0 \\ Y_i^0 \end{bmatrix} = \begin{bmatrix} X_i^0 \\ Y_i^0 \end{bmatrix} - \begin{bmatrix} -D_i \\ -N_i \end{bmatrix}Q_i^0, \quad (10)
\]

† Here, we relax the condition \( (Y_i^0 - N_iQ_i)^{-1} \in H^n \) for simplicity. The relaxation, however, does not affect the main results of the paper. In particular, the condition is always satisfied when \( N_i \) is strictly causal.

![Fig. 1. The ratio \( \zeta(n) \) when \( \delta = 10^{-6} \).](image-url)
which satisfies
\[
\left\| X_i' \right\|_\infty = \left\| X_i^0 \right\|_\infty - \left[ -D_i \right] Q_i^0 \leq \sqrt{\left\| T_v \right\|^2 + 1} \leq \sqrt{\left( \mu_i^2 - 1 + \delta \right)^2 + 1} = \mu_i + (1 + \delta) \frac{\left( \mu_i^2 - 1 + \delta \right)}{\mu_i + \delta}
\]
where \( \mu = \max \{ \mu_1, \mu_2 \} \) and
\[
f(\delta) = \sqrt{\mu_i^2 + \delta(2\mu_i^2 - 1 + \delta) - \mu_i}
\]
with \( \xi(\delta) \) defined in Lemma 1.

Proof. See Appendix A.

To illustrate the improved continuity property, we apply the \( \delta \)-suboptimal design to the same plants in Example 1.

Example 1 (continued). The \( \delta \)-suboptimal design procedure of this section applied to \( P_1 \) and \( P_2 \) in Example 1 produces the \( \delta \)-suboptimal central controllers. Denote the controllers by
\[
(X^\delta_i, Y^\delta_i), \quad i = 1, 2.
\]
Define
\[
\zeta_\delta = \left\| \begin{bmatrix} X^\delta_i \\ Y^\delta_i \end{bmatrix} - \begin{bmatrix} X^\delta \end{bmatrix} \right\|_2.
\]
Let \( \varepsilon_1 \) and \( \varepsilon_2 \) be the same as Example 1.

Figure 2 illustrates \( \zeta_\delta \) as a function of \( n \) when \( \delta = 10^{-2} \) and \( 10^{-3} \). It shows a significant improvement on the Lipschitz continuity.

3.4. Suboptimal design: general plants

When the representation of a plant is not normalized, the following normalization procedure can be performed. Suppose \( P = \mathcal{N}_0(D^0)^{-1} = (D^0)^{-1}\mathcal{N}_0 \) is a coprime factorization (not necessarily normalized) representation. In the subsequent discussions, we will concentrate on the right coprime representations with the understanding that similar results can be easily obtained for the left version. Define
\[
M = (D^0)^*D^0 + (\mathcal{N}_0^*)^*\mathcal{N}_0 \in L^\infty.
\]
Since \( (\mathcal{N}_0, D^0) \) is coprime, by the Operator-Valued Corona Theorem (Nikol’skii, 1986):
\[
\inf_{|z|=1} \left| \mathcal{N}_0(z) \right| > 0
\]

\[\text{Fig. 2. The ratio } \zeta_\delta(n) \text{ when } \delta = 10^{-2} \text{ and } 10^{-3}.\]
where \( \tau_{\text{min}} \) is the smallest singular value, which implies \( M^{-1} \in L^\infty \). As a result, spectral factors \( T \) of \( M \), i.e. outer functions \( T = (H^n)^{x_0} \) satisfying

\[
M = T^*T,
\]

have inverses \( T^{-1} \) in \( (H^n)^{x_0} \). Consequently, the corresponding system \( T \in H^n \) has inverse \( T^{-1} \in H^n \). The spectrum factorization is unique up to a unitary constant matrix in \( C^{nxn} \). It follows that for \( (N, D) \in H^\infty \times H^\infty \) defined by

\[
N = N^0T^{-1}, \quad D = D^0T^{-1},
\]

we have \( P = ND^{-1} \) and

\[
N^*N + D^*D = I.
\]

In other words, \( (N, D) \) is a normalized coprime factorization of \( P \). We should study the Lipschitz continuity from \( H^\infty \) to \( H^2 \) of feedback mappings from \( (N^0, D^0) \) to stabilizing controllers.

**Assumption 2.** For \( i = 1, 2 \), let \( P_i = N_i(D_i)^{-1} - (D_i)^{-1}N_i^0 \) be coprime factorizations and

\[
M_i = (D_i)^*D_i + (N_i)^*N_i^0 \in L^n.
\]

Suppose spectral factors \( T_i \) of \( M_i \) satisfy \( T, T^{-1} \in B \) with

\[
\alpha := \max_{i=1,2} \|T_i\|_B \quad \text{and} \quad \beta := \max_{i=1,2} \|T_i^{-1}\|_B.
\]

Note that since spectral factors of \( M_i \) are unique up to a constant unitary matrix and the norm \( \|\cdot\|_B \) is unitary invariant, \( \alpha \) and \( \beta \) are in fact independent of the choice of spectral factors.

**Lemma 2** (Wang, 1991). If for \( i = 1, 2 \), \( M_i \) satisfies Assumption 2, then there exist spectral factors \( T_i \) of \( M_i \) for which

\[
\|T_2 - T_1\|_\infty \leq \|T_2 - T_1\|_B \leq c \|M_2 - M_1\|_B
\]

where \( c = \alpha \beta^2(1 + \alpha \beta + \alpha^2 \beta^2) \).

The main result of this section is Theorem 2.

**Theorem 2.** Suppose for \( i = 1, 2 \), \( P_i = N_i(D_i)^{-1} = (D_i)^{-1}N_i^0 \) satisfies Assumption 2.

\[
d_0 := \max \left\{ \left( \left\| \frac{N_i}{D_i} - \frac{N_i^0}{D_i^0} \right\|_B, \left\| \frac{N_i}{D_i} - \frac{N_i^0}{D_i^0} \right\|_B \right) \right\}
\]

\[
< \frac{1}{\rho(\delta)(\beta + 2\alpha^2 \beta^2)},
\]

then the \( \delta \)-suboptimal central solutions satisfy

\[
\left\| \begin{bmatrix} X_1 \n \ Y_1 \end{bmatrix} - \begin{bmatrix} X_2 \n \ Y_2 \end{bmatrix} \right\|_2 \leq g(\delta)(\beta d_0 + \alpha \beta c \|M_1 - M_2\|_B)
\]

\[
\leq g(\delta)(\beta + 2\alpha^2 \beta^2)c d_0,
\]

where \( c \) is defined in Lemma 2.

**Proof.** It follows easily from Lemma 2 and Theorem 1. \( \square \)

4. STABILITY ANALYSIS OF SLOWLY VARYING SYSTEMS

Theorem 1 contains a design procedure for slow adaptive stabilization in \( G \). We will start in this section with a discussion on the double algebra framework introduced in Wang and Zames (1989), and establish stability analysis of slowly varying systems within the framework.

4.1. **Local–global double algebra**

For \( K \in G \) with a convolution sum representation

\[
(Ku)(\tau) = \sum_{\theta = -\infty}^{\tau} k(\tau, \theta)u(\theta), \quad \tau \in \mathbb{Z},
\]

the local system of \( K \) at \( \tau \in \mathbb{Z} \) is the (time-invariant) operator \( K_\tau \) with the same domain as \( K \) satisfying

\[
(K_\tau u)(\tau) := \sum_{\theta = -\infty}^{\tau} k(\tau, \tau - (\tau - \theta))u(\theta), \quad \tau \in \mathbb{Z}.
\]

The kernel of \( K_\tau \) will be denoted by \( k_\tau \), and \( k_\tau(\theta) = k(\tau, \tau - \theta), \tau, \theta \in \mathbb{Z} \), and the frequency response of \( K_\tau \) will be \( K_\tau \). The terms local and frozen-time will be used interchangeably.

We define the two products on the space \( G \): (i) the usual composition product, which will be called the ‘global product’, and denoted explicitly by \( \cdot \), although that symbol, as usual, will mostly be suppressed in notation, i.e. \( F \cdot K = FK \); and (ii) a ‘local product’, denoted by \( \otimes \) and defined as follows. For any \( F, K \in G \), \( F \otimes K \) is the unique operator in \( G \) whose local operators satisfy

\[
(F \otimes K)_\tau = F \cdot K_\tau, \quad \forall \tau \in \mathbb{Z}.
\]

A double algebra is any subspace of \( G \) which is
equipped with both products and is an algebra with respect to each one. In particular, the space $\mathcal{B}$ equipped with both products is clearly a double algebra which will also be denoted by $\mathcal{B}$.

For $K \in \mathcal{B}$, $K^{-1}$ will denote its global inverse,

$$K^{-1}K = KK^{-1} = I,$$

and $K^{\circ}$ its local inverse,

$$K^{\circ} \otimes K = K \otimes K^{\circ} = I.$$

The local–global coupling in a double algebra is expressed by the product difference binary operator $\nabla : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ defined by

$$\nabla F = FK - F \otimes K. \quad (16)$$

The (time) variation rate of $K \in \mathcal{B}$ in any given norm $\| \cdot \|$, denoted by $d_{\| \cdot \|}(K)$ is

$$d_{\| \cdot \|}(K) := \| KT - TK \|, \quad (17)$$

where $T$ is the right shift operator. $K$ will be said to commute approximately with the shift if $d_{\| \cdot \|}(K) < \| K \|$.

For $\sigma > 1$, let $\mathcal{E}_\sigma$ be the subspace of $\mathcal{B}$,

$$\mathcal{E}_\sigma := \left\{ K \in \mathcal{B} : \| K \|_{\sigma} := \sup_{r \in \mathbb{Z}} \| K_r \|_{\sigma} < \infty \right\}.$$

$\mathcal{E}_\sigma$ is a Banach space under $\| \cdot \|_{\sigma}$ as norm. For $K \in \mathcal{E}_\sigma$,

$$\| K \|_{\sigma} \leq \kappa_{\sigma} \| K \|_{\sigma},$$

where

$$\kappa_{\sigma} = \frac{\sigma}{\sqrt{\sigma^2 - 1}}.$$

For $p = 2$ or $\infty$, the variation rate of $K$ in $\mathcal{E}_\sigma$, denoted by $d_{\| \cdot \|_{\sigma}}(K)$, is

$$d_{\| \cdot \|_{\sigma}}(K) = \sup_{r \in \mathbb{Z}} \| (KT - TK) \|_{\sigma}.$$  

It is easy to show that

$$d_{\| \cdot \|_{\sigma}}(K) = \sigma \sup_{r \in \mathbb{Z}} \| K_r - K_{r-1} \|_{\sigma}.$$  

Note that

$$d_{\| \cdot \|_{\sigma}}(K) \leq d_{\| \cdot \|_{\infty}}(K).$$  

The following bound on the $\nabla$ operator is obtained in Zames and Wang (1991).

**Proposition 2.** For any $F, K \in \mathcal{E}_\sigma$,

$$\| \nabla FK \|_{\sigma} \leq c_{\sigma} \| F \|_{\sigma} d_{\| \cdot \|_{\sigma}}(K), \quad (18)$$

where

$$c_{\sigma} = \frac{\kappa_{\sigma}}{e \ln \sigma}.$$
Robust stabilization of slowly varying systems

coprime and normalized in $H^\infty_{sa}$, i.e. $N, D, \tilde{N}, \tilde{D} \in E$ and for all $t \in Z$:

\[
\begin{align*}
(N(\sigma z))^* N(\sigma z) + (D(\sigma z))^* D(\sigma z) &= I, \\
\tilde{N}(\sigma z)^* \tilde{N}(\sigma z) + \tilde{D}(\sigma z)^* \tilde{D}(\sigma z) &= I,
\end{align*}
\]

and

\[
\begin{align*}
\tilde{X}^0 N_0 + \tilde{Y}^0 D_0 &= I, \\
\tilde{N}^0 X_0^0 + \tilde{D}^0 Y_0^0 &= I,
\end{align*}
\]

for some $\tilde{X}^0, \tilde{Y}^0, X^0, Y^0 \in E$. Assume

\[
\begin{align*}
\max \left\{ d_{1^+}(\begin{bmatrix} N \\ D \end{bmatrix}), d_{1^+}(\begin{bmatrix} \tilde{N} \\ \tilde{D} \end{bmatrix}) \right\} &\leq \gamma_0 < \infty, \\
\max \{ \|N\|_{(\sigma)}, \|D\|_{(\sigma)}, \|\tilde{N}\|_{(\sigma)}, \|\tilde{D}\|_{(\sigma)} \} &\leq k_0 < \infty.
\end{align*}
\]

For a fixed $\delta > 0$, define

\[
\mu_{\delta} = \sup_{\epsilon > 2} \inf_{Q, \epsilon > 2} \left\{ \left\| X^\epsilon Y^\epsilon \right\|_{1^+} - \left[ \begin{array}{c} -D_0 \\ N_0 \end{array} \right] Q_{1^+} \right\},
\]

\[
k = \max \{ k_0, \mu_{\delta} + f(\delta) \},
\]

\[
\gamma = \gamma_0 \max \{ 1, g(\delta) \},
\]

where $g(\delta)$ and $f(\delta)$ are defined in Theorem 1. Also, define $a$ and $b$ as in Lemma 3.

**Theorem 3.** Let $(X', Y')$ be the local $\delta$-suboptimal central solutions in $E_{sa}$, i.e. for each $t \in Z$, $(X'_t, Y'_t)$ is the $\delta$-suboptimal central solution in $H^\infty_{sa}$ from the data $(N_t, D_t)$ and $(\tilde{N}_t, \tilde{D}_t)$. If

\[
0 < a\gamma < 1 < b\gamma
\]

then

\[
R = \tilde{N}X' + \tilde{D}Y'
\]

has inverse in $B$ and $F = X'(Y')^{-1}$ stabilizes $P$ in $B$.

**Proof.** It follows immediately from Theorem 2 and Lemma 3. \qed

**Remark.** (i) For sufficiently small $\gamma_0$, $0 < a\gamma (1 - b\gamma)^{-1} < 1$ are satisfied, and Theorem 3 claims that the frozen-time $\delta$-suboptimal central solutions provide a (time-varying) stabilizing controller for the slowly varying plant $P$.

(ii) While Theorem 3 is a stability result, the proposed design will certainly guarantee a certain level of performance, say, measured by sensitivity functions. However, since optimal robust performance is not directly incorporated in the frozen-time design, the design procedure needs to be modified if robust performance is to be pursued.

5.2. Robustness of the design

The stabilization achieved by $(X', Y')$ is, in fact, robust in the local gap metric, as shown in Theorem 4.

Define the local directed gap ball $Ball(P, \epsilon)$ as follows:

\[
(i) \text{the center } P = D^{-1}N = ND^{-1} \text{ satisfies Assumption 3 and the hypothesis of Theorem 3;} \\
(ii) \text{every } P_1 \in Ball(P, \epsilon) \text{ admits a coprime factorization representation } P_1 = D_1^{-1}N_1 \text{ with}
\]

\[
\left\| \begin{bmatrix} \tilde{N}_1 \\ \tilde{D}_1 \end{bmatrix} - \begin{bmatrix} \tilde{N} \\ \tilde{D} \end{bmatrix} \right\|_{(\sigma)} < \epsilon,
\]

and

\[
\max \{ \|\tilde{N}\|_{(\sigma)}, \|\tilde{D}\|_{(\sigma)} \} \leq k_0,
\]

\[
d_{1^+}(\begin{bmatrix} \tilde{N} \\ \tilde{D} \end{bmatrix}) \leq \gamma_0.
\]

**Theorem 4.** $F = X'(Y')^{-1}$ constructed in Theorem 3 robustly stabilizes $Ball(P, \epsilon)$ in $B$, provided $\epsilon$ satisfies

\[
0 \leq b_1(\epsilon) := \frac{1}{1 - \epsilon k \|R_P\|_{(\sigma)}} < \infty
\]

and

\[
0 \leq \frac{b_1(\epsilon)a\gamma}{1 - b_1(\epsilon)b\gamma} < 1.
\]

**Remark.** Since

\[
0 \leq \frac{a\gamma}{1 - b\gamma} < 1
\]

and $b_1(\epsilon) \to 1$ as $\epsilon \to 0$, for sufficiently small $\epsilon$, the hypothesis of Theorem 4 is always satisfied.

**Proof.** See Appendix C.

6. CONCLUSIONS

It is shown in this paper that the $\delta$-suboptimal central solutions to robust stabilization in the gap metric have certain desirable Lipschitz continuity properties, which can be employed to derive a frozen-time design procedure for robust stabilization in slowly time-varying systems which admit local coprime factorization representations. Continuity properties of feedback design other than optimal robustness in the gap metric are being investigated. It will also be of interest to study different types of continuity arising in other control design problems.

While the idea of $\delta$-suboptimal designs works for the frozen-time synthesis of $H^\infty$-interpolation as in Wang and Zames (1990) and in this paper, it may not be suitable for other design problems. For instance, problems of $|S| + |T|$ (sensitivity plus complementary sensitivity optimization) can
have discontinuous cost functions, and hence the mapping from plants to controllers cannot be made continuous by this idea. New design approaches need to be pursued in these cases.

As correctly pointed out by an anonymous reviewer, the selection of the central suboptimal solution is merely one possible way of achieving a slowly varying controller. It is certainly possible that there may be other suboptimal designs which produce better slowly varying controllers. A deep understanding of the geometry of the suboptimal gap controllers will be of essential importance in pursuing such controllers.

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REFERENCES


APPENDIX A

Proof of Theorem 1. Since the δ-suboptimal central solution is unique and independent of the choice of (X!, Y!), we can select any solutions to the Bezout equation in the optimization problem (9). In particular, if (X!, Y!) is the δ-suboptimal central solution to

\[ \hat{N}_1 X_1 + \hat{D}_1 Y_1 = I, \]

then by (11)

\[ \left\| \begin{array}{c} X! \\ Y! \end{array} \right\|_\infty \leq \rho(\delta). \]

By the standard small gain argument, since

\[ \| (\hat{N}_1 - N_1) X_1 + (\hat{D}_1 - D_1) Y_1 \|_\infty = \left\| \begin{array}{c} \hat{N}_1 - N_1 \\ \hat{D}_1 - D_1 \end{array} \right\|_\infty \left\| \begin{array}{c} X_1 \\ Y_1 \end{array} \right\|_\infty \leq \rho(\delta), \]

by the hypothesis

\[ N_1 X_1 + D_1 Y_1 = I - ((N_1 - X_1) X_1 + (D_1 - D_1) Y_1), \]

has inverse in H∞ with

\[ \| (\hat{N}_1 X_1 + \hat{D}_1 Y_1) \|_\infty \leq \frac{1}{1 - \rho(\delta)} =: \gamma(\delta). \]

It follows that

\[ \left( X_1 = (\hat{N}_1 X_1 + \hat{D}_1 Y_1)\right)^{-1} = Y_1 (\hat{N}_1 X_1 + \hat{D}_1 Y_1)^{-1} \in H^\infty \times H^\infty, \] (A.2)

is a solution to the Bezout equation

\[ \hat{N}_1 X_2 + \hat{D}_1 Y_2 = I. \]

Furthermore,

\[ \left\| \begin{array}{c} X_2 \\ Y_2 \end{array} \right\|_\infty \leq \rho(\delta) \left\| \begin{array}{c} \hat{N}_1 \\
\hat{D}_1 \end{array} \right\|_\infty \left\| \begin{array}{c} X_1 \\ Y_1 \end{array} \right\|_\infty \leq \rho(\delta) \rho(\delta) d = \rho(\delta)^2 d. \] (A.3)

Consequently, by the definition of V,

\[ \| V_2 - V_1 \|_\infty \leq \| (D^2 X_1 - N^2 Y_1) - (D^2 X_1 - N^2 Y_1) \|_\infty \leq \| D^2 - D_1 X_1 - (N^2 - N^2 Y_1) Y_1 \|_\infty \leq \| N^2 - N^2 Y_1 \|_\infty \left\| \begin{array}{c} X_1 \\ Y_1 \end{array} \right\|_\infty \leq \rho(\delta)^2 \gamma(\delta) d. \] (A.4)
Moreover, from (10) and (A.2),
\[ 0 \leq \left\| \begin{bmatrix} -D_2 \\ N_1 \end{bmatrix} Q_2 \right\|_2 \]
\[ \leq \left\| X_2 \right\| \left\| Y_1 \right\| + \left\| X_2 \right\| \left\| Y_1 \right\| \\
\leq \rho(\delta) + \left\| X_2 \right\| \left\| (\bar{N}_1 X_1 + \bar{D}_2 Y_1)^{-1} \right\|_2 \\
\leq \rho(\delta)(1 + v(\delta), \quad (A.5) \]
by (A.1).

As a result, from the definition of \( X_5 \) and \( Y_5 \), we obtain
\[ \left\| \begin{bmatrix} X_5 \\ Y_5 \end{bmatrix} \right\|_2 \leq \left\| X_2 \right\| \left\| Y_1 \right\| + \left\| X_2 \right\| \left\| Y_1 \right\| \\
+ \left\| X_2 \right\| \left\| (\bar{N}_1 X_1 + \bar{D}_2 Y_1)^{-1} \right\|_2 \\
< \rho(\delta)(v(\delta) + \rho(\delta)(1 + v(\delta))d \\
+ \delta(\delta)(1 + \rho(\delta))d = g(\delta)d, \]
by (A.3), (A.4) and (A.5), and Theorem 1 follows. \( \square \)

APPENDIX B

Proof of Lemma 3.\[ \mathbf{R}_x = \bar{N} \mathbf{X} + \bar{D} \mathbf{Y} = (\bar{N} \mathbf{X} + \bar{D} \mathbf{Y}) \mathbf{R}_x = \mathbf{R}_x \mathbf{R}_y - \mathbf{R}_x \mathbf{V} \mathbf{R}_y, \]

i.e.
\[ \mathbf{R}_y = \mathbf{R}_x \mathbf{R}_y \]

By the small gain argument, since \( \left\| \mathbf{R}_x \mathbf{V} \mathbf{R}_y \right\|_2 < 1, \)
which implies
\[ \left\| \mathbf{R}_x \right\|_2 \mathbf{R}_y - 1 \in \mathbb{B} \]
and
\[ \left\| \mathbf{R}_y \right\|_2 \leq \frac{\left\| \mathbf{R}_y \right\|_2}{1 - \left\| \mathbf{R}_x \mathbf{V} \mathbf{R}_y \right\|_2}. \]

Furthermore, hypothesis \( \left\| (\bar{N} \mathbf{X} + \bar{D} \mathbf{Y}) \mathbf{R}_x \right\|_2 < 1, \)
implies
\[ (1 + (\bar{N} \mathbf{X} + \bar{D} \mathbf{Y}) \mathbf{R}_x)^{-1} \mathbf{R}_x \in \mathbb{B}. \]

Invertibilities (B.1) and (B.3) imply that
\[ \mathbf{R}_x \in \mathbb{B}. \]

Moreover, by (18)
\[ \left\| \mathbf{R}_x \mathbf{V} \mathbf{R}_y \right\|_2 \leq c_d \left\| \mathbf{R}_x \mathbf{V} \right\|_2 \mathbf{V}_2 \mathbf{X}_2 \]
\[ = c_d \left\| \mathbf{R}_x \mathbf{V} \right\|_2 \mathbf{V}_2 \mathbf{X}_2 + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2 \]
\[ \leq c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{X}_2 \right\| \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2 \]
\[ \leq c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2 \]
\[ \leq c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2. \]

As a result, Inequality (19) is satisfied whenever
\[ \left\| \mathbf{R}_x \mathbf{V} \mathbf{R}_y \right\|_2 \leq c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2 < 1, \]
and in this case,
\[ \left\| \mathbf{R}_x \right\|_2 \left\| \mathbf{R}_y \right\|_2 \leq \frac{c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2}{1 - c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2}. \]

In addition,
\[ \left\| (\bar{N} \mathbf{X} + \bar{D} \mathbf{Y}) \mathbf{R}_x \right\|_2 \leq \frac{c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2}{1 - c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2}. \]

APPENDIX C

Proof of Theorem 4. For every \( \mathbf{P}_i = \bar{D}_i \mathbf{N}_i = \text{Ball}(\mathbf{P}, \epsilon), \)
\[ \mathbf{R}_x = \bar{N}_i \mathbf{X} + \bar{D}_i \mathbf{Y} = \bar{N}_i \mathbf{X} + \bar{D}_i \mathbf{Y} + \bar{D}_i \mathbf{Y} + (\bar{N}_i \mathbf{N}_i \mathbf{X} + (\bar{D}_i \mathbf{D}_i \mathbf{Y}) = \mathbf{R}_x + \bar{N}_i \mathbf{N}_i \mathbf{X} + (\bar{D}_i \mathbf{D}_i \mathbf{Y}) \mathbf{R}_x = \mathbf{R}_x + \mathbf{B}_x \mathbf{B}_y \]

Since \( \mathbf{R}_x \mathbf{V} \mathbf{R}_y \in \mathbb{F}_x \) and
\[ \left\| \mathbf{R}_x \mathbf{V} \mathbf{R}_y \right\|_2 \leq \frac{c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2}{1 - c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2} \]

Furthermore, following the same argument as in (B.4) and (B.6), we obtain
\[ \left\| \mathbf{R}_x \mathbf{V} \mathbf{R}_y \right\|_2 \leq c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2 \leq \frac{c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2}{1 - c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2} \]

Also,
\[ \left\| (\bar{N}_i \mathbf{X} + \bar{D}_i \mathbf{Y}) \mathbf{R}_x \right\|_2 \leq \frac{c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2}{1 - c_d \left\| \mathbf{V} \right\|_2 \left\| \mathbf{Y}_2 \right\| + \left\| \mathbf{Y}_1 \right\| \mathbf{Y}_1 \mathbf{Y}_2} \]

It follows from (C.2) and (C.4) and Lemma 3 that
\[ \left( \bar{N}_i \mathbf{X} + \bar{D}_i \mathbf{Y} \right)^{-1} \in \mathbb{B} \]
and \( \mathbf{F} = (\mathbf{Y}^\mathbf{y})^{-1} \) stabilizes \( \mathbf{P}_i \) in \( \mathbb{B} \). Since \( \mathbf{P}_i \in \text{Ball}(\mathbf{P}, \epsilon) \) is arbitrary, \( \mathbf{F} \) robustly stabilizes \( \text{Ball}(\mathbf{P}, \epsilon) \) in \( \mathbb{B} \). \( \square \)