Disturbance Attenuation via State Feedback for Systems with a Saturation Nonlinearity in the Control Channel*

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Key Words—$H^\infty$ control; state feedback; nonlinear systems; stability; saturation.

Abstract—We study the problem of disturbance attenuation via state feedback for asymptotically stable systems with a saturation nonlinearity in the control channel. The disturbance is assumed to be bounded, with the bound known. This amounts to adding a saturation nonlinearity in the disturbance channel. A sufficient condition is derived in terms of algebraic matrix inequalities, which are much simpler to solve than the general Hamilton-Jacobi-Isaacs partial differential inequality. A suboptimal control law is given based on the solution to the Riccati inequalities. The computational complexity of solving the optimal controller is similar to that of linear $H^\infty$ controller design.

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1. Introduction

This article is concerned with the problem of disturbance attenuation via state feedback for systems with a saturation nonlinearity in the control channel. The control of such systems has been studied by many researchers using different approaches. A great deal of work has been done on stability analysis and stabilization of such systems (Boyd et al., 1994). While stability is one of the most important issues in control, often it is also required that the closed-loop systems satisfy certain performance criteria. During the past few years, there have been some major developments in $H^\infty$ control of nonlinear systems (see e.g. Başar and Bernhard, 1991; Van der Schaft, 1992; Isidori and Astolfi, 1992; Ball et al., 1993; Isidori and Kang, 1993; Pan and Başar, 1993) following the breakthrough in the linear $H^\infty$ theory, where the time-domain analysis resulted in elegant state-space formulas for the solution of the standard linear $H^\infty$ control problem in terms of the solutions of the two Riccati equations (Doyle et al., 1989). The results in Willems (1972) and Hill and Moylan (1980) were instrumental to the development of the time-domain approach of nonlinear $H^\infty$ control. In the $H^\infty$ control design, in addition to stability, the closed-loop system is required to satisfy the $H^\infty$ performance criterion. As pointed out in Ball et al. (1993), $H^\infty$ design has its unique features and is advantageous in many situations. While some neat theoretical results have appeared, unlike its linear counterpart, most of the results on $H^\infty$ control for nonlinear systems involve solving partial differential inequalities or equations. For this reason, it is usually extremely difficult to actually design a controller using the existing results on $H^\infty$ control for nonlinear systems. Also, most results on nonlinear $H^\infty$ control are local. Recently there has been increasing interest in reducing the complexity of the conditions involved in nonlinear $H^\infty$ design and obtaining global results. In Helton and Zhan (1995) the $H^\infty$ control problem was considered in which the nonlinearities take some simple forms such as piecewise-linear functions and saturation in the control channel. In these special cases, the general conditions could be somewhat simplified. In fact, some sufficient conditions were given for solvability, which only involves algebraic inequalities, in contrast to the partial differential inequalities in the general case. This shows that the design is quite different from the linear design, and much more difficult to solve numerically. In Zhan et al. (1995), the problem of $H^\infty$ control for systems with sector-bound nonlinearities was considered. The energy function (Ball et al., 1993) was assumed to have a special form in order to obtain practical sufficient conditions. Inspired by these developments in nonlinear $H^\infty$ control, in this article, we study the $H^\infty$ control problem for systems with saturation in the control channel. Our goal is to obtain sufficient conditions that are numerically easy to check (at least easier than those in Helton and Zhan, 1995). The result should be valid globally or in a large region in state space.

Because of the limitation in the magnitude of the control signal, it would be almost impossible to guarantee the system performance if the disturbance were not bounded, unless the open-loop system itself (i.e. the system with zero input) satisfied the performance criteria or somehow one could guarantee that the trajectories of the closed loop system stayed inside a relatively small neighborhood of the origin. To guarantee that the trajectories stay inside a given set for arbitrary disturbance is certainly not easy. Here we propose to consider a more practical case where the magnitude of the disturbance is bounded from above by a known positive number over the time period of interest, i.e. $|w(t)| \leq w_0$, $\forall t \in [0, T]$. We shall see that this assumption amounts to adding a saturation nonlinearity in the disturbance channel.

This key assumption allows us to get a firm result which is valid globally. It is worth noting that the main contribution of this article is on $H^\infty$ performance. The stability result comes as a by-product, and could be somewhat more conservative than those obtained in Lin et al. (1994) and Liu et al. (1993).

The article is organized as follows. In Section 2, we introduce the notation used, and present some well-known results on $H^\infty$ control for nonlinear systems, which will be the starting point for our main result. The latter is presented in Section 3. Conclusions are drawn in Section 4.

2. Preliminaries

In this article, we shall more or less follow the notation used in Ball et al. (1993). $R^n$ denotes the $n$-dimensional Euclidean space of real numbers. For a vector $v \in R^n$, we denote its transpose by $v^T$. For a differentiable function $e(v)$ of $v$, $\nabla e(v)$ denotes its gradient, namely

$$\nabla e(v) = \frac{de}{\partial v_1} \ldots \frac{de}{\partial v_n}^T.$$

The following function will be used:

$$\text{Sat}[u] = \begin{cases} 
1 & \text{if } u > 1, \\
u & \text{if } -1 \leq u \leq 1, \\
-1 & \text{if } u < -1.
\end{cases}$$

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Consider a nonlinear system
\[ \dot{x} = f(x, u, w), \]
\[ y = C(x), \]  
(1)
where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) the control input, \( w \in \mathbb{R}^p \) the disturbance signal and \( y \in \mathbb{R}^q \) the variable to be penalized; \( f(x, u, w) \) and \( C(x) \) are smooth nonlinear mappings with compatible dimensions and satisfy \( f(0) = 0 \) and \( C(0) = 0 \). The disturbance attenuation problem can be formulated as follows.

Find a feedback control law \( u = k(x) \) such that, under the zero initial condition \( x(0) = 0 \),
\[ \int_0^\infty w^T(s)W_1w(s)\,ds \geq \int_0^\infty \left[ y^T(s)W_1y(s) + u^T\text{Sat}\{u/u_o\}\right]\,ds, \]  
(5)
and, when \( w = 0 \), the equilibrium point \( x = 0 \) is globally asymptotically stable.

\begin{proof}
By Theorem 2.1 in Ball et al. (1993), (2) follows from Definition 5.4. Note that one of the following conditions holds
\[ (i) \quad u_o < w_oW_1; \]
\[ \text{and} \quad \text{H}_{2}(w_{<u_{o}}, u_{+}) < 0, \]  
(6)
Choose \( e(x) = x^T \mathbf{1} \) as a Lyapunov function of the closed-loop system with \( w = 0 \). The equilibrium point \( x = 0 \) is globally asymptotically stable.

The performance requirement \( H_2(w, u) < 0 \) is apparently equivalent to \( H_2(w_{\geq u_o}, u_{o}) < 0 \), where \( w_{\geq u_o} = w_o \text{Sat}\{B^TEx/(w_oW_1)\} \). And the optimal control law is given by \( u = -B^TEx \).
\end{proof}
systems are globally asymptotically stable, the stability result similar to the famous Riccati inequality in linear algebraic Riccati-type matrix inequalities, which are very matrix inequalities (LMI), for which commercial software is matrix inequalities can be easily put into equivalent linear same as that of designing a linear stable plants. The condition in Theorems 3.1 involves is conservative. One can easily derive from the Raccati on these facts, it is fair to say that, as far as numerical nonlinearity in the control channel. Although the closed-loop article is on 4.

Therefore, the complexity of our control design is the linear control problem.

As we mentioned earlier, the main contribution of this article is on H performance of systems with a saturation nonlinearity in the control channel. Although the closed-loop systems are globally asymptotically stable, the stability result is conservative. One can easily derive from the Riccati inequalities in (i) or (ii) of Theorem 3.1 that the matrix A is asymptotically stable. Therefore our result applies only to stable plants. The condition in Theorems 3.1 involves algebraic Riccati-type matrix inequalities, which are very similar to the famous Riccati inequality in linear H control theory. Therefore the complexity of our control design is the same as that of designing a linear H controller. These matrix inequalities can be easily put into equivalent linear matrix inequalities (LMI), for which commercial software is available to obtain numerical solutions. Interested readers are referred to Boyd et al. (1994) for details on LMI. Based on these facts, it is fair to say that, as far as numerical solutions are concerned, solving DAP is as easy as solving the standard linear H control problem.

4. Conclusions

We have derived a sufficient condition, under a practical assumption |w| < w0 on the disturbance, for the solvability of the disturbance attenuation problem using state feedback for systems with a saturation nonlinearity in the control channel. The condition is easy to check using numerical methods. The computational complexity is at the same level as those in linear H control design. The resulting suboptimal controller is given in terms of the solution to certain Riccati inequalities.

The general theory of nonlinear H control involves solving partial differential inequalities or equations for positive-definite or semidefinite functions. In a sense, the nonlinear H control theory is very similar to the Lyapunov stability theory, where one needs to solve a partial differential inequality V(x) < 0 for V(x) > 0. As we all know, the importance of Lyapunov stability theory is not that it provides a practical way of solving the partial differential inequality, but rather that it can be used to study stability for special nonlinear systems when practical conditions can be obtained. In a series of efforts, we have been trying to study systems with special nonlinearities such as piecewise-linear function, saturation nonlinearity in the control channel and sector-bound nonlinearities. We strongly believe that this is an important issue in nonlinear H control that should be further investigated.

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References


