Enhanced feedback robustness against communication channel
multiplicative uncertainties via scaled dithers

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ABSTRACT

In this paper, a new method is introduced to enhance feedback robustness against communication
gain uncertainties. The method employs a fundamental property in stochastic differential equations to
add a scaled stochastic dither under which tolerable gain uncertainties can be much enlarged, beyond
the traditional deterministic optimal gain margin. Algorithms, stability, convergence, and robustness
are presented for first-order systems. Extension to higher-dimensional systems is further discussed.
Simulation results are used to illustrate the merits of this methodology.

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1. Introduction

Recent advance in communication technologies and networked systems in mobile agents, unmanned vehicles, parallel computing, intelligent vehicle systems, tele-medicine, and smart grids has generated intensified research efforts on integrated feedback systems with communication channels [1–3]. The basic control configuration in such systems involves a plant with local sensors and actuators and a remote controller, which are interconnected by communication channels.

Communication channels introduce some unique challenges to feedback systems. Traditionally, uncertainties from communication channels are dominantly modeled as additive noise [4,5]. Since additive noises will not directly affect feedback stability, such pursuit is often concentrated on signal estimation accuracy and noise attenuation. However, advanced communication schemes encounter uncertainties of nonlinear or multiplicative nature. After a signal is sampled, quantized, coded, and transmitted, it propagates through multiple pathways, depending on terrain conditions, buildings, weather conditions, echoes, interferences, and correlations with other signals. They are then collected at the receiver, combined, and decoded. Such a scenario is better represented by random variations on transmission gains whose values can vary over a large range and may change signs as well. Feedback robustness against such gain uncertainties is the focus of this paper.

Feedback stability and robustness have been pursued for channel latency (time delays), packet losses, quantization errors, and other related communication scenarios [6–8]. Minimum channel capacities of noisy communication channels for a feedback system to stabilize an unstable plant are investigated in [6]. Control-oriented communication design, including data compression, quantization, and coding schemes, is studied in an integrated control and communication framework [7]. Presented in [8] are solutions to output variance minimization of systems involving Gaussian channels in the feedback loop. Furthermore, channel delays are treated in [9] by accommodating queuing/buffering times in communication hubs. The optimal stochastic control methodologies are used in an LQG (Linear–Quadratic–Gaussian) problem with delay statistics [8,10]. Complexity issues in networked system identification are studied in [11,12].

There are fundamental limitations of feedback systems on gain robustness, especially sign changes. This paper introduces a new method of using scaled stochastic dithers in communication schemes to enhance feedback robustness beyond the traditional
deterministic optimal gain margin. A dither is an intentionally added noise-like random signal. Traditionally noises are viewed as adversary elements that need to be attenuated or removed. However, random noises can also be used to overcome some system limitations. For example, a quantizer limits greatly information on a signal. However, by adding a well-chosen random signal (a dither) before quantization, far more information on the signal can be recovered [12]. This paper is a new attempt to use random dithers to enhance robustness of feedback systems.

The main idea is based on a fundamental property in Itô’s formula for stochastic differential equations in which the diffusion term affects convergence differently than the drift term. This distinct feature indicates that if a scaled dither is used in transmitting a signal, it may be immune to value or sign changes in transmission gains. This feature and inherent feedback robustness can potentially extend feedback robustness to a much expanded gain uncertainty set. This paper explores algorithms, stability, convergence, and robustness in this framework.

The key idea that the Brownian motion term in stochastic differential equations has the unique square term due to Itô’s formula is well known in the stochastic analysis community and is a classical result [13,14]. However, this term does not appear naturally in control systems. For example, in convergence analysis of stochastic approximation or adaptive filtering algorithms under random noises, the limit obeys an ODE (ordinary differential equation), rather than an SDE (stochastic differential equation). Possibly for such reasons, the unique stabilizing effect of the Brownian motion has not been actually employed in control community as a tool for robustness. Also, the classical notion of gain margins traced back to the time of Nyquist and Bode and was stated in deterministic systems. The idea that randomness in signals can enhance a deterministic robustness requires a new way of thinking.

As a first attempt in this direction, the scope of this paper is limited to first-order systems. For such basic systems, the impact of added dithers on gain margins can be completely characterized. Also, the key ideas are easy to convey in such simplified settings. While extension of this idea to higher dimensional systems is viable, technical details and constraints are more sophisticated. For these reasons, this paper provides some discussions on such potential extensions without concrete results.

The rest of the work is organized in the following sections. Section 2 describes system models and configurations. The main methodology of scaled dithers is also introduced. In Section 3, the main results of the paper are presented. The theoretical foundation of the scaled dither methodology is first established by using the limit SDE method. By using the features of the scaled dither, we show that the feedback robustness ranges can be extended to a larger set involving sign changes. Explicit robustness bounds are established for first-order systems. Section 4 briefly describes how scaled dithers can be used to enhance robustness of typical observers. Extension of this approach to higher-order systems is discussed in Section 5 with a case study and some basic ideas. The paper concludes with some remarks in Section 6.

2. Preliminaries

The basic idea of this paper can be explained from a unique feature of the scalar stochastic differential equation

\[ dx = axdt + bdxw \]

where \( w \) is a standard Brownian motion. By Itô’s formula, the stability (in probability) of this system is determined by \( a - b^2/2 \). In that sense, the Brownian motion term provides a stabilizing action [13–15]. However, such a stochastic term does not occur naturally in systems where observation noises are additive. Also, it is not clear what is the real benefit of using this feature when this system can be easily stabilized by a regular deterministic feedback. Probably for these reasons, this distinctive feature of stochastic systems has never been used to enhance robustness beyond what can be achieved from deterministic feedback control.

This paper explores potential enhancement of robustness against multiplicative uncertainties in feedback loops by using this feature. We study this in a networked control setting that involves communication channels. We show that a Brownian motion term can be generated by adding a scaled dither and it can enlarge gain margins that cannot be achieved by deterministic feedback.

2.1. Systems

The basic feedback system consists of a plant whose output is processed and communicated through a dedicated communication link to form a feedback loop. The plant \( P(s) \) and controller \( C(s) \) are combined to form the open-loop system \( G \) that has a state-space realization

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t).
\end{align*}
\]

Without uncertainties from communication channels, the feedback loop is formed by the negative unity feedback \( u = -y \) and the resulting closed-loop system is

\[
\dot{x}(t) = Ax(t) + Bu(t) = (A - BC)x(t) = A_0x(t).
\]

When communication channels are involved, the output signal \( y(t) \) will be sampled. Suppose that \( t_k \) is the \( k \)th sampling interval which may change with time. For small \( t_k \), the open-loop system is approximated by

\[
\begin{align*}
x_{k+1} &= x_k + t_k(Ax_k + Bu_k) \\
\dot{y}_k &= Cx_k
\end{align*}
\]

where starting at \( t_0 = 0 \) with \( t_k = \sum_{i=1}^{k} \tau_i \), we denote \( x_k = x(t_k) \) and \( y_k = y(t_k) \). The feedback control is \( u_k = -y_k \). Under the standard zero-order hold (ZOH) framework, \( u(t) = u_k, t \in [t_k, t_{k+1}) \).

Typically, observation noises and communication uncertainties are limited to additive noises. The characterizing feature of additive noises is that they do not depend on signals. If an uncertainty depends on the signal itself, it becomes multiplicative type. Multiplicative uncertainties occur in systems and affect system stability and performance. For example, sensor magnifications, actuator mappings, and signal propagation fading are typical multiplicative uncertainties. Depending on communication schemes, digital communications involve many function blocks, such as sampling, data compression, quantization, source coding, channel coding, and modulation at the sending side, and demodulation, decoding, and signal reconstruction at the receiving side [4]. Consequently, communication channels introduce broader uncertainties of various types, including those that are signal dependent. In this paper, we consider combined additive and multiplicative communication uncertainties

\[
\dot{y}_k = g_0y_k + e_k,
\]

where \( e_k \) is an additive noise and \( g_0 \) is the gain uncertainty, both being random. Since the additive noise \( e_k \) is independent of the signal \( y_k \), it will affect system performance, such as control accuracy and error bounds, but not robust stability. With the feedback control \( u_k = -\dot{y}_k = -g_0y_k - e_k \), the closed-loop system becomes

\[
x_{k+1} = x_k + t_k(A - g_0BC)x_k - Be_k.
\]

Note that for constant uncertain gains \( g_0 = g \), stability of the closed-loop system is determined by \( \dot{A} - gBC \).
The random gain \(g_k\) will affect stability directly. The robustness of a feedback system against gain uncertainties is often quite limited. For example, consider the open-loop system
\[ \dot{x}(t) = ax(t) + bu(t) \]
with \(a > 0\) and \(b > 0\). This system can be stabilized by a constant feedback \(u = -gx\), if \(a - bg < 0\). The robustness range for the uncertain gain \(g\) is \((a/b, \infty)\). Obviously, the feedback mechanism cannot tolerate sign changes in transmission gains.

2.2. Scaled dithers

Instead of sending only \(y_k\), a scaled dither is now added to form a new signal \(z_k\) to be sent through the communication channel
\[ z_k = y_k + \alpha(t_k, y_k)d_k, \quad (5) \]
where \(d_k\) is the stochastic dither. The scaling factor \(\alpha(t_k, y_k)\) is both signal dependent and sampling interval dependent, and selected as
\[ \alpha(t_k, y_k) = \frac{\gamma}{\sqrt{t_k}}y_k, \quad (6) \]
for some design variable \(\gamma > 0\). The reason for this choice will become clear soon. The communication channel introduces uncertainties as in (3) and generates a received signal \(\hat{z}_k\)
\[ \hat{z}_k = g_kz_k + e_k. \quad (7) \]

Then the feedback becomes \(u_k = -\hat{z}_k\).

**Assumption 1.** (1) \(\{d_k\}\) is an independent and identically distributed (i.i.d.) Gaussian distributed random dither such that \(Ed_k = 0\) and \(Ed_k^2 = 1\).

(2) The unknown gain \(\{g_k\}\) is a bounded sequence of stationary, uniform mixing process [16, pp. 350–351], independent of \(\{d_k\}\) such that its mixing measure \(\psi_k\) satisfies \(\sum_{k=1}^{\infty} \psi_k < \infty\) and that \(Eg_k = \bar{g}\).

(3) \(\{e_k\}\) is another sequence of stationary mixing process such that \(Ee_k = 0\) and \(E|e_k|^{1+q} < \infty\) for some \(q > 0\), and that its mixing measure \(\psi_k\) satisfies \(\sum_{k=1}^{\infty} \psi_k^{\frac{1}{1+q}} < \infty\).

From (2) and (7), the control signal is
\[ u_k = -g_k \left( y_k + \frac{\gamma}{\sqrt{t_k}}y_k d_k \right) - e_k \\
= -g_k \left( Cx_k + \frac{\gamma}{\sqrt{t_k}}C_k d_k \right) - e_k. \quad (8) \]

Consequently, the closed-loop system becomes
\[ x_{k+1} = x_k + \tau_k (A - g_kBC)x_k - \sqrt{\tau_k g_k} \gamma BCx_k d_k - \tau_k Be_k. \quad (9) \]

where the sampling interval sequence \(\{\tau_k\}\) is interpreted interchangeably as the stepsize, and assumed to satisfy \(\tau_k > 0\), \(\tau_k \to 0\) as \(k \to \infty\), and \(\sum_{j=0}^{k=\infty} \tau_j = \infty\). In applications of periodic sampling, \(\tau_j\) is a small constant \(\tau\). So we will also consider a constant stepsize algorithm of the form
\[ x_{k+1} = x_k + \tau (A - g_kBC)x_k - \sqrt{\tau g_k} \gamma BCx_k d_k - \tau Be_k. \quad (10) \]

For simplicity, assume that \(x_0\) is not random and not dependent on \(\tau\).

Algorithms in (10) can be studied by stochastic approximation methods [17]. To relate them to continuous-time dynamic systems, for the decreasing stepsize algorithm (9), we define \(t_k\) as before, introduce piecewise constant interpolations \(x^\prime(t)\) as \(x^\prime(t) = x_k\) for \(t \in [t_k, t_{k+1})\), and denote the shifted sequence of functions \(x^\prime(t) = x^\prime(t + t_k)\), and \(m(t) = \max(k : t_k \leq t)\). For the constant stepsize algorithm (10), we define \(x^\prime(t) = x_k\) for \(t \in [rk, rk + \tau)\).

Consider, for instance, the case of constant stepsize algorithm and the scaled noise
\[ \tilde{w}^\prime(t) = \sqrt{\tau} \sum_{k=0}^{t/\tau - 1} g_k d_k, \]
where \(t/\tau = \lfloor t/\tau \rfloor\) is the integer part of \(t/\tau\) (for notational simplicity, we suppress the floor function notation henceforth). Under Assumption 1, as \(\tau \to 0\), \(\tilde{w}^\prime(\cdot)\) converges weakly to \(\tilde{w}(\cdot)\), a Brownian motion with covariance \(g^2\tau\) and
\[ \tilde{g}^2 = Eg_k^2 d_k^2 \overset{\infty}{\to} 2 \sum_{k=1}^{\infty} Eg_k d_k g_0 d_0 = Eg_0^2. \]

Likewise, we can define for the algorithm (9)
\[ \tilde{w}^\prime(t) = \sum_{j=k}^{m(t)+\tau - 1} \sqrt{\tau_k} g_k d_j. \]

We can also show that \(\tilde{w}^\prime(\cdot)\) converges weakly to a Brownian motion \(\tilde{w}(\cdot)\) with covariance \(g^2\tau\).

The Brownian motion limits obtained above can be represented by using the above observations and the techniques of stochastic approximation. Under constant step sizes, \(x^\prime(\cdot)\) converges weakly to \(x(\cdot)\) that is a solution of a stochastic differential equation (SDE). In this process, the noise \(e_k\) and the signal \(g_k\) vary much faster than that of the “state” \(x\). As a result, \(e_k\) is averaged to 0, and the drift involving \(g_k\) is averaged to \(\bar{g} = Eg_k\). Furthermore, the Brownian motion \(\tilde{w}(\cdot)\) (or \(\tilde{w}^\prime(\cdot)\)) can be replaced by a standard Brownian motion \(w(\cdot)\). The proof of the following theorem is omitted and the reader is referred to [17, Chapters 7 and 10] for further details. Consequently, the stability of (9) or (10) can be analyzed by using its limit SDE.

**Theorem 1.** Under Assumption 1, both \(x^\prime(\cdot)\) and \(x^\prime(\cdot)\) converge weakly to \(x(\cdot)\) such that \(x(\cdot)\) is a solution of the stochastic differential equation
\[ dx = (A - \bar{g} BC)xdt + \bar{g} \gamma BCdw, \quad (11) \]
where \(w(\cdot)\) is a standard Brownian motion.

3. Feedback robustness against gain uncertainties

3.1. Stochastic differential equations and Itô’s formula

The enhancement of stability robustness by the scaled dither is based on Itô’s Formula in stochastic differential equations [13, 18, 19]. In its applications to linear time-invariant systems, suppose that \(x(t) \in \mathbb{R}^n\) is a real-valued stochastic process satisfying
\[ x(t) = x(t_0) + \int_{t_0}^{t} Mx(r)dr + \int_{t_0}^{t} Hx(r)dw(r), \quad (12) \]
where \(M, H \in \mathbb{R}^{n \times n}\) and \(w(\cdot)\) is the one-dimensional standard Brownian motion. The solution \(x(\cdot)\) can also be written as
\[ dx = Mxdt + Hxdw. \quad (13) \]

In our approach, the diffusion is created by the added scaled dither.

**Definition 1.** The SDE (13) is said to be exponentially stable w.p.1 if its Liapunov exponent satisfies
\[ \lim_{t \to \infty} \frac{1}{t} \log |x(t)| < 0 \quad \text{w.p.1}, \]
where \(| \cdot |\) is the Euclidean norm.
From (11), we have \( M = A - \bar{g} BC \) and \( H = \bar{g} \gamma BC \), with \( \bar{g} = Eg_k \) and \( \bar{g} = E \delta_k^2 \). For the case of scalar systems, \( x \) is a scalar and
\[
dx = mxdt + hdxw.
\]
By Itô’s Formula [13, 19], the solution to (14) is
\[
x(t) = e^{(m - \frac{1}{2} h^2)t + hw} x(0),
\]
with the given initial condition \( x(0) \). By the local martingale convergence theorem [20], \( u(t) / t \to 0 \) w.p.1. As a result,
\[
\lim_{t \to \infty} \sup_{t \in \Omega} \frac{\log |x(t)|}{t} = m - \frac{1}{2} h^2.
\]
Consequently, the SDE (14) is exponentially stable if \( m - \frac{1}{2} h^2 < 0 \). The dither term \( -\frac{1}{2} h^2 \) provides a stabilizing effect.

### 3.2. Impact of the scaled dither on gain robustness

To proceed, we now explore first-order systems in detail. In this case, \( A = a, B = b, C = c \), all scalar constants. For the system to be controllable and observable, it requires that \( b \neq 0 \) and \( c \neq 0 \). By (16) with \( m = a - \bar{g} bc \) and \( h = \bar{g} \gamma bc \), the stability condition becomes
\[
f_2(\bar{g}, \bar{g}^2) = m - \frac{1}{2} h^2 - a - \bar{g} bc - \frac{1}{2} \bar{g}^2 \gamma^2 b^2 c^2 < 0.
\]
Suppose that the uncertainty on \( g_k \) is characterized by an uncertainty set \( \Omega \) on \( (\bar{g}, \bar{g}^2) \). Then the robust stability requires that
\[
\sup_{(\bar{g}, \bar{g}^2) \in \Omega} f_2(\bar{g}, \bar{g}^2) < 0,
\]
or equivalently
\[
\sup_{(\bar{g}, \bar{g}^2) \in \Omega} (\bar{g}^2 \gamma^2 b^2 c^2 + 2 \bar{g} bc - 2a) > 0.
\]
By expressing \( g_0 = \bar{g} + \epsilon \) where \( E \epsilon = 0 \) and \( E \epsilon^2 = \sigma_k^2 \), we have
\[
\bar{g}^2 = Eg^2 = \bar{g}^2 + \bar{g}^2.
\]
Hence, the condition (17) is equivalent to
\[
\gamma^2 b^2 c^2 + 2 \bar{g} bc - 2a + \sigma_k^2 \gamma^2 b^2 c^2 > 0.
\]

**Theorem 2.** Suppose that \( \sigma_k^2 \) is bounded below by some constant \( \mu \), \( \sigma_k^2 \geq \mu > 0 \). If \( \gamma \) is designed to satisfy
\[
\gamma^2 > \frac{a + \sqrt{a^2 + \mu b^2 c^2}}{\mu b^2 c^2}
\]
then the SDE (14) is exponentially stable for all \( \bar{g} \) and \( \sigma_k^2 \geq \mu \).

**Proof.** The roots of the polynomial (as a function of \( \bar{g} \))
\[
\bar{g}^2 \gamma^2 b^2 c^2 + 2 \bar{g} bc - 2a + \sigma_k^2 \gamma^2 b^2 c^2 = 0
\]
are
\[
\lambda_{1,2} = \frac{-bc \pm |bc| \sqrt{1 + 2a \gamma^2 - \sigma_k^2 \gamma^4 b^2 c^2}}{\gamma^2 b^2 c^2}.
\]
Observe that the condition for \( \lambda_{1,2} \) to be complex is
\[
1 + 2a \gamma^2 - \sigma_k^2 \gamma^4 b^2 c^2 < 0.
\]
By solving \( \gamma^2 \) from
\[
1 + 2a \gamma^2 - \sigma_k^2 \gamma^4 b^2 c^2 = 0
\]
we obtain the positive solution as
\[
\gamma^2 = \frac{a + \sqrt{a^2 + \sigma_k^2 b^2 c^2}}{\sigma_k^2 b^2 c^2}.
\]
Since the right hand side of (24) is monotone with respect to \( \sigma_k^2 \), if (21) is satisfied,
\[
1 + 2a \gamma^2 - \sigma_k^2 \gamma^4 b^2 c^2 < 0.
\]
This implies that \( \lambda_{1,2} \) are complex. Consequently, (20) is satisfied. This implies that the SDE (14) is exponentially stable. Since this is valid for all \( \bar{g} \) and any \( \sigma_k^2 \geq \mu \), the proof is complete.

In the special case of deterministic but unknown \( g_k \), namely \( \sigma_k^2 = 0 \), the above analysis can be directly applied to the degenerative stability condition
\[
\bar{g}^2 \gamma^2 b^2 c^2 + 2 \bar{g} bc - 2a > 0.
\]
In this case the following results hold.

**Theorem 3.** (1) If \( a < 0 \), representing stable open loop systems, then by selecting \( \gamma^2 > 1/(2|a|) \), the closed-loop system is stable for all \( \bar{g} \).

(2) If \( a \geq 0 \), representing unstable open loop systems, then for any given \( \gamma \), the closed-loop system is stable for all \( \bar{g} \in \Omega = (-\infty, \lambda_1) \cap (\lambda_2, \infty) \), where
\[
\lambda_1 = \frac{-bc + |bc| \sqrt{1 + 2a \gamma^2}}{\gamma^2 b^2 c^2}, \quad \lambda_2 = \frac{-bc + |bc| \sqrt{1 + 2a \gamma^2}}{\gamma^2 b^2 c^2}.
\]

**Proof.** Note that the roots of the polynomial \( \gamma^2 \bar{g}^2 b^2 c^2 + 2 \bar{g} bc - 2a \) are
\[
\lambda_{1,2} = \frac{-bc \pm |bc| \sqrt{1 + 2a \gamma^2}}{\gamma^2 b^2 c^2}.
\]
(1) If \( a < 0 \) and \( \gamma^2 > 1/(2|a|) \), \( \lambda_1 \) and \( \lambda_2 \) are complex. As a result, \( \gamma^2 \bar{g}^2 b^2 c^2 + 2 \bar{g} bc - 2a > 0 \) for all \( g_k \). This implies that the SDE (14) is exponentially stable for all \( \bar{g} \).

(2) If \( a \geq 0 \), then \( 1 + 2a \gamma^2 \geq 0 \). It follows that \( \gamma^2 \bar{g}^2 b^2 c^2 + 2 \bar{g} bc - 2a > 0 \) if and only if \( g_k < \lambda_1 \) or \( \bar{g} > \lambda_2 \).

**Remark 1.** Note that \( \lim_{\gamma \to \infty} \lambda_1 = 0, \lim_{\gamma \to \infty} \lambda_2 = 0 \). As a result, for any compact set \( \Omega \subseteq (-\infty, 0) \cap (0, \infty) \), there exists \( \gamma \) such that the closed-loop system is robustly stable for all \( g_k \). The added dither creates a desirable stabilizing factor that can tolerate random uncertain gains with sign changes. Such robustness cannot be achieved by a deterministic feedback.

### 3.3. Robustness bounds on relative gain uncertainties

It is common in practice that gain uncertainties are expressed in relative terms: \( g_k = (1 + \delta_k) \bar{g} \), where \( \bar{g} \) is the nominal gain and \( \delta_k \) is the relative gain uncertainty.

**Assumption 2.** \( \delta_k \) is i.i.d. with \( E \delta_k = 0 \) and \( E \delta_k^2 = \sigma_k^2 > 0 \).

Under **Assumption 2**, \( E \bar{g} = \bar{g} \) and \( E \bar{g}^2 = \bar{g}^2 (1 + \sigma_k^2) \). In this case, the stability condition (19) takes the form
\[
\bar{g}^2 (1 + \sigma_k^2) \gamma^2 b^2 c^2 + 2 \bar{g} bc - 2a > 0.
\]
(25)
The following results hold. While the results cannot be directly derived from **Theorem 3**, the proof is similar, and hence omitted.
Fig. 1. Comparison between a deterministic feedback and a feedback with a stochastic dither: the left-side plots show feedback without added dithers. (a) Top: nominal feedback. The closed-loop system is stable. (b) Middle: the gain is perturbed from 3 to 1. The closed-loop system is unstable. (c) Bottom: the sign of the gain is changed to negative. The closed-loop system is unstable. The right-side plots show feedback with an added dither of $\gamma^2 = 4$. (a) Top: nominal feedback. The closed-loop system is stable. (b) Middle: the gain is perturbed from 3 to 1. The closed-loop system remains stable. (c) Bottom: the sign of the gain is changed to negative. The closed-loop system remains stable.

**Theorem 4.** (1) If $a < 0$, representing stable open-loop systems, then by selecting $\gamma^2 > 1/(2|a|)(1 + \sigma_\gamma^2)$, the closed-loop system is stable for all $\tilde{g}$. 
(2) If $a \geq 0$, representing unstable open-loop systems, then for any given $\gamma$, the closed-loop system is stable for all $\tilde{g} \in \Omega = (-\infty, \lambda_1) \cap (\lambda_2, \infty)$, where

$$\lambda_1 = -bc - |bc|\sqrt{1 + 2a\gamma^2(1 + \sigma_\gamma^2)} / \gamma^2(1 + \sigma_\gamma^2)b^2c^2,$$

$$\lambda_2 = -bc + |bc|\sqrt{1 + 2a\gamma^2(1 + \sigma_\gamma^2)} / \gamma^2(1 + \sigma_\gamma^2)b^2c^2.$$

**Example 1.** Consider the system $\dot{x} = 2x + u$. We compare the closed-loop systems with or without the added dithers. We consider the deterministic unknown gains $g_0$. Suppose that the gain uncertainty satisfies $|g_0| \geq 1$. Fig. 1 shows three cases: (a) nominal feedback with $g_0 = 3$; (b) the gain is perturbed to a much reduced value $g_0 = 1$; (c) when channel uncertainties result in a sign change on the gain to $g_0 = -1$. Without the dither, the closed-loop system is unstable under (b) and (c). The simulation results demonstrate that with the added dither, the feedback system retains stability under all perturbed gains, hence is more robust.

3.4. Pure dither feedback

It is possible to use a pure dither feedback to gain robust stability. Suppose that instead of (5), we only use $z_k = \frac{1}{\sqrt{2}}\mathbf{d}_k$. Then, the SDE (14) becomes

$$\dot{x} = ax dt + \tilde{g}_\gamma b c dw.$$

The stability condition is simplified to

$$a - \frac{1}{2}\gamma^2b^2c^2 < 0.$$

**Theorem 5.** Suppose that for some constant $\mu > 0$,

$$\gamma^2 > \frac{2|a|}{\mu b^2c^2}.$$

Then the closed-loop system is robustly stable for all $\tilde{g}^2 \geq \mu > 0$.

**Proof.** Under (26), if $\tilde{g}^2 \geq \mu > 0$

$$a - \frac{1}{2}\gamma^2b^2c^2 < a - \frac{1}{2}\mu\gamma^2b^2c^2 < 0.$$

This implies stability. □

**Remark 2.** In some sense, the condition $\tilde{g}^2 \geq \mu > 0$ is necessary. If $\tilde{g}^2 = 0$, then the communication channel is disconnected with probability one. In this case the feedback is running in open loop. So, if the open-loop system is unstable, feedback stability is lost, regardless what feedback control is used.

4. Robust state observers

In this section, we briefly describe the potential usage of stochastic dithers to achieve robustness in state observers. Standard full-order observers, Luenberger observers, Kalman filters assume the complete knowledge of the system parameters in their designs. In general, they are not robust with respect to model uncertainties.

Consider a first-order system

$$\begin{align*}
\dot{x} &= ax + bu \\
y &= cx.
\end{align*}$$

For simplicity, we assume $b = 1$ (just group $bu$ as the new $u$) and $c = 1$ (just group $y/c$ as $y$). Hence,

$$\begin{align*}
\dot{x} &= ax + u \\
y &= x.
\end{align*}$$
For a meaningful discussion on stabilization, we assume $a > 0$ (so the open-loop system is unstable). The standard state estimators assume the full knowledge of system parameters.

**Luenberger observers:**

For this basic system, the Luenberger observer will simply use

$$\hat{\chi} = y$$

as a reduced order observer. Now, suppose that there is a multiplicative uncertainty of transmitting $y$ with

$$\hat{\chi} = \hat{y} = gy$$

and $g$ is an uncertainty. Consequently, the Luenberger observer becomes

$$\hat{\chi} = \hat{y} = gy.$$ 

If $g \neq 1$, then the Luenberger observer will fail to get the correct state estimate. In other words, it is not robust to such multiplicative uncertainties. If this state estimator is used for feedback design, we have $u = -Kx$. This leads to a closed-loop system

$$\dot{x} = (a - Kg)x.$$ 

(30)

Apparently, if $g$ can assume both positive and negative values, there exists no feedback gain $K$ that can robustly stabilize the system. In other words, the Luenberger observer is fundamentally non-robust in this specific sense.

Now, let us add a dither to the transmission line. Following the same development as in Section 2 that leads to (11), the resulting system is modified from (30) to a stochastic differential equation

$$dx = (a - Kg)x dt + g\gamma Kx dw,$$

(31)

where $w(t)$ is a standard Brownian motion. Consequently, the results of Theorems 3 and 4 are applicable. In other words, by appropriate selection of $y$, stability of (31) will be guaranteed for a much larger range of gain uncertainty on $g$.

**Full-order observers:**

Next, we will try the full-order observer which involves a feedback mechanism. The observer structure is

$$\begin{cases} \dot{x} = a\hat{x} + u - L(\hat{y} - y) \\ \dot{y} = \hat{x} \end{cases}$$

(32)

where $\hat{y} = gy$ and $g$ is the gain uncertainty. Let the state estimation error be $\epsilon = \hat{x} - x$. The error dynamics can be easily derived as

$$\dot{\epsilon} = (a - L)e - L(1 - g)x$$

(33)

which has an additional term due to gain uncertainty.

Next, we design a state feedback $u = -K\hat{x}$. It can be derived that

$$\dot{\epsilon} = ax - L(e + x) = (a - K)x - Ke.$$ 

The overall system dynamics become

$$\begin{cases} \dot{x} = (a - K)x - Ke \\ \dot{\epsilon} = (a - L)e - L(1 - g)x. \end{cases}$$

For stability, $L$ and $K$ are designed to be $a - L = -\lambda_1, a - K = -\lambda_2$ with $\lambda_1 > 0$ and $\lambda_2 > 0$. So, we have

$$\begin{cases} \dot{x} = -\lambda_2 x - Ke \\ \dot{\epsilon} = -\lambda_1 e - L(1 - g)x. \end{cases}$$

(34)

The system matrix is

$$M = \begin{bmatrix} -\lambda_2 & -K \\ -L(1 - g) & -\lambda_1 \end{bmatrix}.$$ 

Its characteristic polynomial is

$$(s + \lambda_1)(s + \lambda_2) - KL(1 - g) = s^2 + (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2 - KL(1 - g).$$

For robust stability, we must have $\lambda_1\lambda_2 - KL(1 - g) > 0$, or equivalently,

$$g > 1 - \frac{\lambda_1\lambda_2}{KL}.$$ 

(36)

Since $\lambda_1 = L - a, \lambda_2 = K - a$, and $a > 0$, we have $0 < 1 - \frac{\lambda_1\lambda_2}{KL} < 1$. Consequently, the condition (36) will always be violated if $g$ can take negative values. In other words, regardless how $K$ and $L$ are designed, the robustness with respect to the gain uncertainty cannot tolerate sign changes on $g$.

It is easy to see that Kalman filters will have the same robustness issues as the full-order observers.

**Using stochastic dithers to enhance observer robustness:**

Without going into too much technical details, we note that adding stochastic dithers amounts to introduce a diffusion term. By suitable choices of dithers and their locations, the observer/feedback system will be modified to a stochastic differential equation in the form of

$$d \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} -\lambda_2 \\ -L(1 - g) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} dt + \sum_{i=1}^2 B_i \begin{bmatrix} x \\ e \end{bmatrix} dw_i,$$

(37)

where for each $\ell = 1, 2$, $w_i$ is a standard scalar Brownian motion, $B_i$ is a suitable $2 \times 2$ matrix, and $w_1$ and $w_2$ are independent. We note that

$$\text{tr} M = \text{tr} \begin{bmatrix} -\lambda_2 & -K \\ -L(1 - g) & -\lambda_1 \end{bmatrix} = -\lambda_1 - \lambda_2 < 0.$$ 

By [21, Theorem 2.2, p. 458] (see also [15]), the asymptotic stability in probability of (37) can be achieved if and only if $\text{tr} M < 0$. Since this condition is satisfied here, robust observers can be achieved. Detailed analysis and more general higher dimensional cases will be treated in a forthcoming paper.

5. Discussions on scaled dithers for higher-dimensional systems

The scaled dithers are effective in providing enhanced robustness in first-order systems. Extension of this idea to higher-dimensional systems can also be beneficial, but requires caution. This is due to more complicated stability conditions and impact of the diffusion term on stability. In general, adding a dither without careful assessment of system structures may destabilize a stable system. However, in the important area of networked consensus control, when properly designed, adding scaled dithers will enhance robust stability. This will be reported in a separate paper [22].

A complete investigation of such scenarios is beyond the scope of this paper. In this section, we use an example to demonstrate effects of adding dithers in improving robustness in consensus control.

5.1. A case study

The constrained consensus was introduced in [23] and applied to several application problems such as power systems and platoon coordination in [24,25]. A networked system consists of $r$ node states denoted by $x_n = [x_n^1, \ldots, x_n^r]^T$. At the control step $n$, the state will be updated from $x_n$ to $x_{n+1}$ by the amount $u_n$

$$x_{n+1} = x_n + u_n$$

(38)

with $u_n = [u_{n1}, \ldots, u_{nr}]^T$. The node subsystems are linked by a network, represented by a directed graph $\tilde{G}$ whose element $(i,j)$ indicates a connection between node $i$ and node $j$, namely estimation of.
the state $x_i$ by node $i$ via a communication link. Skipping derivation details, the state updating algorithm leads to the dynamic equation

$$x_{n+1} = x_n + \mu_n (Mx_n + Wd_n)$$

where the matrices $M$ and $W$ are determined by the network topology and $d_n$ represents additive observation noises.

To be concrete, consider a power management problem in microgrids. A five-bus grid has transmission lines between Buses 1 and 2, 2 and 3, 3 and 4, and 4 and 5, shown in Fig. 2(a). The initial per-unit load distributions on the buses are not balanced with $x_0 = [0.1, 0.2, 0.3, 0.4, 0]^T$. By suitable selection of positive link gains, we have

$$M = \begin{bmatrix} -0.6 & 0.6 & 0 & 2 & 0 \\ 0.6 & -1.8 & 1.2 & 0 & 0 \\ 0 & 0.2 & -2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$W = \begin{bmatrix} 0.3 & -0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9 & 0.9 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

In this case, the eigenvalues of $M$ are $-6.0125, -3.2432, -1.5016, -0.4426, 0$. Since all eigenvalues (except the single eigenvalue at 0) are stable, the control achieves the weighted consensus, as shown in Fig. 2(b).

5.2. Stability analysis

For multidimensional systems (12), we can employ the idea from Khasminskii [26] to examine the stability of the SDE (13). Define the normalized state $x(t)$ and $Mx(t)$ as follows:

$$x(t) = \frac{\xi(t)}{|\xi(t)|}$$

$$Mx(t) = -\frac{\mu(t) (Mx(t) + Wd(t))}{|\xi(t)|}$$

By Itô’s Formula,

$$d\xi(t) = \left[ M\xi(t) - \mu(t) M\xi(t) \right] dt$$

$$+ \left[ -\xi(t) \mu(t) + \frac{1}{2}[M\xi(t)]^2 \right] d\xi(t)$$

$$+ \left( M \xi(t) - \frac{1}{2}[M\xi(t)]^2 \right) dw(t).$$

Let $H = [h_i]$. Denote

$$q_i(x) = \sum_{l_1, l_2} h_{l_1} h_{l_2} x_{l_1} x_{l_2}, \quad i, j = 1, \ldots, n$$

and $Q(x) = [q_i(x)]$. Define

$$\rho(t) = \ln |\rho(t)|$$

As stated in [26, pp. 220–221], since $\xi(t)$ is a diffusion on $\mathbb{S} = \{\xi : |\xi| = 1\}$ (the unit sphere), $\xi(t)$ is ergodic with a unique invariant measure $P(\cdot)$ if

$$\xi^{\top} Q(\xi) \xi = (\xi^{\top} H\xi)^2 \geq K_0 |\xi|^{2}.$$  \hspace{1cm} (40)

By Itô’s Formula,

$$d\rho(t) = \left[ \xi^{\top}(t) M\xi(t) + \frac{1}{2} |H\xi(t)|^2 \right] dt$$

$$+ \xi^{\top}(t) H\xi(t) dw(t).$$

We obtain the following result.

**Theorem 6.** Assume that (40) is satisfied. Let

$$\lambda_0 = \int_{\mathbb{S}} \left[ \frac{\xi^{\top} M\xi}{2} + \frac{1}{2} (|H\xi|^2 - 2 |\xi^{\top} H\xi|^2) \right] P(\xi).$$

where $P(\cdot)$ is the invariant measure. Then the linear SDE (13) is almost surely exponentially stable (resp., unstable) if and only if $\lambda_0 < 0$ (resp., $\lambda_0 > 0$).
The proof of the theorem is omitted. Some details of the proof can be found in [26] for diffusion processes and in [27] for switching diffusion processes. The stability condition (41) is reduced to (17) for first-order systems, which can be analyzed directly. In general, however, the condition (41) needs to be verified numerically.

6. Concluding remarks

This paper introduces the approach of adding scaled dithers to expand robustness capabilities of feedback systems. The approach is introduced in feedback systems with communication channels which involve gain uncertainties including possible sign changes. It is shown that adding a state and sampling-rate dependent dither can enhance feedback robustness beyond the optimal gain margin in deterministic systems. A more comprehensive study of feasibility and limitations of this method is of interest. Utility of this method in systems involving random delays and phase shifts is currently under investigation.

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