High Precision Phase Measurement Using Adaptive Sampling

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Abstract—The conventional phase measurement techniques, described in [1]–[6], introduce error in the phase when the input signals are distorted by harmonics. A new technique, known as adaptive sampling, for high precision phase measurement is introduced in this paper. A digital signal processing approach is used in this new technique. The maximum sampling rate required for this technique is \(k + 2\) samples/cycle of the input signals, i.e., \((k + 2) f_{\text{s}}\) samples/s, where \(k\) is the highest harmonic present in the signals and \(f_{\text{s}}\) is the fundamental frequency of the signals. This sampling rate is way below the Nyquist sampling rate (more than \(2f_{\text{s}}\) samples/s) when \(k\) is a large number. In the adaptive sampling technique, the sampling rate is increased from 3 samples/cycle and then it is gradually increased until the phase is correctly measured. This new phase measurement technique has been verified using synthesized signals.

I. INTRODUCTION

IN CONVENTIONAL techniques [4]–[6] the phase angle between two signals of the same frequency is computed by converting the signals into two square waves and then measuring the time difference either between the zero crossing points or between the pulse centers of the square waves [1], [3]. The conventional techniques introduce error in phase measurement when the input signals are distorted by harmonics. A detailed error analysis of phase measurement of distorted signals is shown in [1].

This paper presents a high precision phase measurement technique using adaptive sampling. In order to measure the phase difference between the fundamental components of two signals, the sampling rate is initially made equal to 3 samples/cycle (\(3f_{\text{s}}\) samples/s). Here, it is assumed that \(f_{\text{s}}\) is a known quantity. If \(f_{\text{s}}\) is an unknown quantity then it could be determined in real-time by measuring the time difference between two successive positive zero crossing points of the signals, provided that at least one signal does not have more than two zero crossing points per cycle. A zero crossing point is considered as a positive zero crossing point where the signal changes from a negative to a positive value. The value of \(f_{\text{s}}\) must be known for the cases when both signals may have more than two zero crossing points per cycle. Actually this will be a rare case (if any), because our objective is to measure the phase difference between two signals when the signals may be distorted by harmonics. Unless the amplitudes of the harmonics are very high (comparable with the fundamental amplitude) there will not be more than two zero crossing points per cycle. In the adaptive sampling technique, if at any time the measured phase is found to be invalid, then the sampling rate is increased by one sample/cycle and the phase is measured again at the new sampling rate. The process of increasing the sampling rate (adaptive sampling) is continued until the phase is accepted as a valid phase or until the sampling rate becomes equal to the maximum sampling rate of the system. The technique which is used to determine whether the measured phase is or is not valid is explained in Section II.

II. HIGH PRECISION PHASE MEASUREMENT TECHNIQUE

Any periodic signal \(x(t)\) can be expressed by a Fourier series representation as

\[
x(t) = a_0 + \sum_{n=1}^{k} a_n \cos(2\pi n ft) + \sum_{n=1}^{k} b_n \sin(2\pi n ft)
\]

\[= a_0 + \sum_{n=1}^{k} c_n \cos(2\pi n f t - \phi_n)
\]

where

\[
c_n = \sqrt{a_n^2 + b_n^2}
\]

and

\[
\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right).
\]

In (1), \(f\) is the fundamental frequency, \(k\) is the highest harmonic present in signal \(x(t)\), \(a_0\) is the dc component of the signal, \(c_n\), and \(\phi_n\), \(1 \leq n \leq k\), respectively represent the amplitude and phase of the \(n\)th harmonic. The Fourier coefficients \(a_n\) and \(b_n\) can be computed as

\[
a_n = \frac{2}{T} \int_{0}^{T} x(t) \cos(2\pi n ft) \, dt
\]

(2a)

\[
b_n = \frac{2}{T} \int_{0}^{T} x(t) \sin(2\pi n ft) \, dt
\]

(2b)

where \(T = 1/f\) is the period of signal \(x(t)\). For a discrete system, (2) can be expressed as

\[
a_n^d = K \sum_{i=0}^{N-1} x(i\Delta T) \cos(2\pi n f i \Delta T)
\]

(3a)

\[
b_n^d = K \sum_{i=0}^{N-1} x(i\Delta T) \sin(2\pi n f i \Delta T)
\]

(3b)
where $K$ is the proportionality constant, $\Delta T$ is the sampling interval, and $N = T/\Delta T$ is the number of samples taken in every cycle of signal $x(t)$. The superscript $d$ indicates that the coefficients are for a discrete system. Applying the theory of signal processing [7], it could be easily proved that the relationship between the coefficients of the analog and discrete systems is

$$a_n^d + jb_n^d = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \left( a_{n+mN} + jb_{n+mN} \right). \quad (4)$$

Fig. 1(a) shows the frequency response of the periodic signal $x(t)$. Fig. 1(b) and 1(c) show the frequency responses of the discrete signal $x(i\Delta T)$, $0 \leq i \leq N - 1$. In Fig. 1(b), $N > 2h$ but in Fig. 1(c), $N < 2h$. The sampling rate in Fig. 1(c) is less than the Nyquist sampling rate. Thus the aliasing effect is seen on the frequency response shown in Fig. 1(c). The phase associated with the $n$th harmonic of signal $x(t)$ can be accurately measured from the samples $x(i\Delta T)$, $0 \leq i \leq N - 1$, if the ratios $b_n^d/a_n^d$ and $b_n/a_n$ are the same. These ratios will be the same if there is no aliasing effect on the $n$th harmonic response. If all the harmonics $i$, $2 \leq i \leq h$, are present in signal $x(t)$, then from Fig. 1(c) it is clear that the $n$th harmonic response will not be affected by the aliasing effect if $(2\pi - 2\pi h/N) > 2\pi n/N$, i.e., if $N > h + n$. Thus the $n$th harmonic phase could be accurately determined if the number of samples per cycle of $x(t)$ is more than $h + n$. Hence, for measuring the fundamental phase ($n = 1$) the sampling rate must be higher than $h + 1$ samples per cycle of $x(t)$, if $x(t)$ has all the harmonics $i$, $2 \leq i \leq h$. This sampling rate could be even lower if some of the harmonics $i$, $2 \leq i \leq h - 1$, are not present in $x(t)$. The required sampling rate could be determined using the adaptive sampling technique developed in this paper and explained later on.

Let us first develop the theory for adaptive sampling. From (1) we can derive the following equation:

$$x(t - t_i) = a_0 + \sum_{n=1}^{h} a_n \cos (2\pi nf(t - t_i))$$

$$+ \sum_{n=1}^{h} b_n \sin (2\pi nf(t - t_i))$$

$$= a_0 + \sum_{n=1}^{h} a_n(\theta) \cos (2\pi nf\theta)$$

$$+ \sum_{n=1}^{h} b_n(\theta) \sin (2\pi nf\theta) \quad (5)$$

where

$$a_n(\theta) = a_n \cos (n\theta) - b_n \sin (n\theta) \quad (6a)$$

$$b_n(\theta) = a_n \sin (n\theta) + b_n \cos (n\theta) \quad (6b)$$

$$\theta = 2\pi ft_i. \quad (6c)$$

The terms $a_n(\theta)$ and $b_n(\theta)$ are the Fourier coefficients of the analog signal $x(t - t_i)$. The relationship between the Fourier coefficients of $x(t - t_i)$ and the corresponding coefficients of the discrete signal $x(i\Delta T - t_i)$, $0 \leq i \leq N - 1$, is the same as the relationship shown by (4).

From now on we will use the notation $\phi_n(\theta, k_1, k_2, \cdots)$ to represent the $n$th harmonic phase of the discrete signal $x(i\Delta T - t_i)$, $0 \leq i \leq N - 1$, when the $n$th harmonic response is affected by the harmonics $k_1, k_2, \cdots$ due to the aliasing effect. We will also use the notation $\phi_n(0, 0, 0, \cdots)$ to represent the $n$th harmonic phase of analog signal $x(t)$. If the sampling rate, $N$, is greater than $h + n$ samples/cycle then we can write

$$a_n^d(\theta) + jb_n^d(\theta) = \frac{1}{\Delta T} (a_n(\theta) + jb_n(\theta)) \quad (7)$$

and the phase of the $n$th harmonic response of the discrete signal $x(i\Delta T - t_i)$, $0 \leq i \leq N - 1$, can be expressed as

$$\phi_n(\theta, 0, 0, \cdots)$$

$$= \tan^{-1} \left( \frac{b_n^d(\theta)}{a_n^d(\theta)} \right)$$

$$= \tan^{-1} \left( \frac{a_n(\theta) \sin (n\theta) + b_n \cos (n\theta)}{a_n(\theta) \cos (n\theta) - b_n \sin (n\theta)} \right)$$

$$\{ \text{using (6) and (7)} \}$$

$$= \phi_n(0, 0, 0, \cdots) + n\theta. \quad (8)$$

Our adaptive sampling technique is developed based on the condition given in (8). This is a necessary (but not sufficient) condition for the response of the $n$th harmonic
of a discrete signal to be free from aliasing effects. The main objective of this paper is to find a sufficient condition for checking whether or not the response is free from aliasing effects at a lower sampling rate.

If the sampling rate is less than \( h + n \) samples/cycle then the \( n \)th harmonic response of the discrete signal \( x(i \Delta T) \), \( 0 \leq i \leq N - 1 \), may be affected by other harmonics (say harmonics \( k_1, k_2, \cdots \)). Thus the \( n \)th harmonic phase of the discrete signal \( x(i \Delta T) \), \( 0 \leq i \leq N - 1 \), can be expressed as

\[
\phi_n(0, k_1, k_2, \cdots) = \tan^{-1} \left( \frac{b_0^d}{a_0^d} \right)
\]

\[
= \tan^{-1} \left( \frac{b_n + b_{k_1} + b_{k_2} + \cdots}{a_0 + a_{k_1} + a_{k_2} + \cdots} \right) \quad \text{using (4)}
\]

\[
= \tan^{-1} \left( \frac{\sum b_i}{\sum a_i} \right),
\]

where \( i = n, k_1, k_2, \cdots \). (9)

The \( n \)th harmonic phase of the discrete signal \( x(i \Delta T - t_i) \), \( 0 \leq i \leq N - 1 \), can be expressed as

\[
\phi_n(\theta, k_1, k_2, \cdots) = \tan^{-1} \left( \frac{b_0^d(\theta)}{a_0^d(\theta)} \right)
\]

\[
= \tan^{-1} \left( \frac{\sum b_i(\theta)}{\sum a_i(\theta)} \right)
\]

\[
= \tan^{-1} \left( \frac{\sum (a_i \sin(i\theta) + b_i \cos(i\theta))}{\sum (a_i \cos(i\theta) - b_i \sin(i\theta))} \right),
\]

where \( i = n, k_1, k_2, \cdots \). (10)

Using (10) and (11) we can show that the condition \( \phi_n(\theta, k_1, k_2, \cdots) = \phi_n(0, k_1, k_2, \cdots) + n\theta \) is upheld if the following condition is true:

\[
\left( \sum a_i \cos(n\theta) - b_i \sin(n\theta) \right) 
\cdot \left( \sum a_i \sin(i\theta) + b_i \cos(i\theta) \right) 
= \left( \sum a_i \sin(n\theta) + b_i \cos(n\theta) \right) 
\cdot \left( \sum a_i \cos(i\theta) - b_i \sin(i\theta) \right)
\]

where \( i = n, k_1, k_2, \cdots \). (12)

Equation (12) can be written in reduced form as follows:

\[
\sum_j \left( D_j \sin(i\theta - n\theta) + E_j \left[ 1 - \cos(i\theta - n\theta) \right] \right) = 0
\]

(13)

where

\[
D_j = \left( \sum a_j \right) a_i + \left( \sum b_j \right) b_i,
\]

for \( j = n, k_1, k_2, \cdots \)

\[
E_j = \left( \sum b_j \right) a_i - \left( \sum a_j \right) b_i,
\]

where \( i = k_1, k_2, \cdots \).

Let us assume that (13) has the following \( r \) nonzero roots in the range \( 0 < \theta < 2\pi \):

\[
\theta = \psi_i
\]

where

\[
i = 1, 2, 3, \cdots, r
\]

and

\[
0 < \psi_i < 2\pi.
\]

(14)

Thus \( \phi_n(\theta, k_1, k_2, \cdots) \) would be equal to \( \phi_n(0, k_1, k_2, \cdots) + n\theta \) if \( t_i \) in (5) is such that \( \theta \) satisfies (14). Actually this is the case which we would like to avoid, because in this case the measured phase \( \phi_n(0, k_1, k_2, \cdots) \) would be erroneously taken as the value of \( \phi_n \). Thus our objective is to select \( \theta \) in such a way that it does not satisfy (14), and for this value of \( \theta \) the condition \( \phi_n(\theta, k_1, k_2, \cdots) = \phi_n(0, k_1, k_2, \cdots) + n\theta \) would be satisfied if and only if \( k_1 = k_2 = k_3 = \cdots = 0 \), i.e., if and only if the \( n \)th harmonic response is not affected by any other harmonics.

If the value of \( t_i \) or \( \theta \) is selected from the output of a random number generator which generates random numbers with uniform distribution in the range \( 0 < \theta < 2\pi \), then it is very unlikely that during every trial, \( \theta \) will be equal to \( \psi_i (i = 1, 2, 3, \cdots, r) \). We can also select \( \theta \) from a predetermined list of nonzero numbers. The num-
number of elements in the list must be greater than \( r \). The phase could be measured using the algorithm shown in Table I. The fundamental phase \( \phi_k \) can be measured by setting the value of \( n \) to 1 in the algorithm shown in Table I. The phase difference between the fundamental components of two signals \( x_1(t) \) and \( x_2(t) \) can be measured by synchronously executing the phase measurement algorithm, and then taking the difference of the fundamental phase of the two signals.

A frequency multiplier is necessary to implement the adaptive sampling technique. The frequency multiplier presented in [9] can generate the output pulses very accurately, where the maximum error in the position of the output pulses is one period of the master clock. If the frequency multiplier is fabricated in a single IC chip, then a very high frequency (at least 50–60 MHz) master clock could be used, and as a result the error introduced by this frequency multiplier would be almost negligible if we want to measure the phase for the signals of frequency under 100 KHz. A preliminary hardware implementation of the adaptive sampling technique is presented in [10].

There are many well-known techniques for generating random numbers. The technique shown in [8] generates 65 536 random numbers without repeating any previously generated number. A hardware random number generator can be designed using a 16-bit shift register and a 2-input exclusive OR gate (see Fig. 2). The shift register performs a right shift operation only, and its initial content must be a nonzero number. This hardware random number generator will generate 65 535 nonzero random numbers without repeating any number. Such a random number generator could be used to generate \( \theta \). In order to determine the phase with 100-percentage reliability, the value of TRIAL\(_{\text{max}}\) in the algorithm must be greater than the number of nonzero roots, \( r \), of (13). The value of \( r \) will depend on the maximum absolute value of \( (i - n) \), where, \( i = k_1, k_2, k_3, \ldots \). If the \( n \)th harmonic response of discrete signal \( x(i \Delta T) \), \( 0 \leq i \leq N - 1 \), is affected by the highest negative harmonic, then the maximum value of \( |i - n| \) is \( h + n \), where \( h \) is the highest harmonic present in signal \( x(t) \). In this case the number of nonzero real roots, \( r \), of (13) can be expressed as \( r \leq 2(h + n) - 1 \) (see Appendix A for proof). For any given application of this phase measurement technique, if it is known that \( h_{\text{max}} \) is the highest harmonic that can ever occur in signal \( x(t) \), then the value of TRIAL\(_{\text{max}}\) can be expressed as TRIAL\(_{\text{max}} = 2(h_{\text{max}} + n) \), which is greater than the number of nonzero roots of (13). The difference between \( h \) and \( h_{\text{max}} \) is that, \( h \) is the highest harmonic present in \( x(t) \), but \( h_{\text{max}} \) is the highest harmonic that can ever occur in \( x(t) \). If \( h_{\text{max}} \) is an unknown quantity, then a low-pass filter with a very high cutoff frequency (say \( f_c \)) can be used to filter \( x(t) \) before the phase is measured. The minimum value of the cutoff frequency, \( f_c \), must be such that the filtering operation does not change the \( n \)th harmonic phase, which is to be measured. If a low-pass filter is used then \( h_{\text{max}} = f_c / f_s \), where \( f_s \) is the fundamental frequency of signal \( x(t) \). In Section III a method has been suggested to measure phase with the value of TRIAL\(_{\text{max}} = 2 \). If this method is used then a low-pass filter is not necessary even when \( h_{\text{max}} \) is unknown. However, this measurement will be at the cost of momentary reliability (explained in Section III).

It is 100 percent guaranteed that if \( \theta \) (a nonzero quantity) is selected TRIAL\(_{\text{max}}\) times (where TRIAL\(_{\text{max}} = 2(h_{\text{max}} + n) \)) from a random number generator or from a list having TRIAL\(_{\text{max}}\) nonzero elements, then there will be at least one value of \( \theta \) which is not a root of (13). For this value of \( \theta \), the condition \( \phi_k(\theta, k_1, k_2, \ldots) = \phi_\theta(0, k_1, k_2, \ldots + n \theta) \) would be satisfied if and only if the \( n \)th harmonic response of \( x(i \Delta T) \), \( 0 \leq i \leq N - 1 \) is not affected by any other harmonics of \( x(t) \). Hence, \( \phi_k(0, k_1, k_2, \ldots) \) can be accepted as the value of \( \phi_\theta(0, 0, 0, \ldots) \), i.e., \( \phi_\theta \).

### Table I

**Phase Measurement Algorithm for nth Harmonic Phase**

1. Sampling rate \( N = n + 2 \)
2. Get \( N \) samples from one cycle of \( x(t) \)
3. Using equation (3) find \( b_0^R \) and \( b_0^I \), and then compute \( \phi_{0}(0, k_1, k_2, \ldots) = \tan^{-1}\left(\frac{b_0^I}{b_0^R}\right) \)
4. Trial Number TRIAL = 0
5. Select a non-zero \( \theta \) from a random number generator or from a predetermined list.
6. TRIAL = TRIAL + 1 (increment trial number)
7. (a) Get \( N \) samples from one cycle of \( x(t) \), where \( t_k = k(2\pi\theta/n) \).
   (b) Using equation (12) find \( b_0^R(0) \) and \( b_0^I(0) \), and then compute \( \phi_\theta(0, k_1, k_2, \ldots) = \tan^{-1}\left(\frac{b_0^I(0)}{b_0^R(0)}\right) \)
8. If \( |\phi_\theta(0, k_1, k_2, \ldots) - |\phi_{0}(0, k_1, k_2, \ldots + n\theta)| > \epsilon \) (\( \epsilon \) is a tolerance limit)
   Then \( N = N + 1 \)
   If \( N > S_{\text{MAX}} \) (\( S_{\text{MAX}} \) is the maximum sampling rate (samples/sec))
      Go To Step 15
   Else Go To Step 2.
   Else if TRIAL = TRIAL\(_{\text{MAX}}\) (TRIAL\(_{\text{MAX}}\) is the maximum number of trials)
      Go To Step 9.
   Else Go To Step 5.
9. Accept \( \phi_\theta(0, k_1, k_2, \ldots) \) as the value of the \( n \)th harmonic phase \( \phi_\theta \). Continue the measurement process by repeating steps 5 and 7 only, and dynamically compute the average value of the phase. If the phase at any instance is found to be too different (i.e., the difference is greater than \( \epsilon \)) from the average value
   Then \( N = N + 1 \)
   If \( N > S_{\text{MAX}} \)
      Go To Step 10.
   Else Go To Step 2.
10. STOP. Phase cannot be measured, because the signal \( x(t) \) is distorted by a very high harmonic.

### III. Performance of the Algorithm

In Section II it has been shown that if TRIAL\(_{\text{MAX}} = 2(h_{\text{MAX}} + n) \), then the phase could be measured with 100-percentage reliability provided the measurement process stops at step 9 of the algorithm. In this section an expression will be derived for the average number of trials that will be required for a sampling rate to figure out that the sam-
pling rate is not high enough to compute the phase. An expression for the average time to compute phase (phase acquisition time) will also be divided.

Let us assume that the random number generator generates $N_R$ numbers before repeating its sequence. We can say that $k$ trials will be required to measure the phase, if during the first $(k - 1)$ trials the random number generator generates $(k - 1)$ nonzero roots of (13) and during the $k$th trial it generates a number which is not a root of (13). Thus the probability that $k$ trials will be required can be expressed as

$$P(k) = \frac{r}{N_R} \cdot \frac{r - 1}{N_R - 1} \cdot \cdots \cdot \frac{r - (k - 2)}{N_R - (k - 2)} \cdot \frac{N_R - r}{N_R - (k - 1)}, \quad \text{for } k = 1, 2, \ldots, r + 1$$

where $r = 2(h_{\text{max}} + n) - 1$ is the maximum number of nonzero roots of (13). The average number of trials that will be required for any sampling rate to determine that the sampling rate is not high enough to measure the phase can be computed as

$$\text{TRIAL}_{\text{avg}} = \sum_{k=1}^{r+1} k \cdot P(k).$$

(16)

It is very difficult to find a closed form expression for the term $\text{TRIAL}_{\text{avg}}$. However, an upper limit of the value of $\text{TRIAL}_{\text{avg}}$ could be determined as follows. Let us define a term $Q(k)$ as

$$Q(k) = \left(\frac{r}{N_R}\right)^{k-1}, \quad \text{for } k = 1, 2, 3, \ldots, r + 1.$$  

(17)

Since $N_R > r$, $Q(k) > P(k)$ for $k = 1, 2, 3, \ldots, r + 1$. Hence, we can say that $\text{TRIAL}_{\text{avg}} < T_{\text{max}}$, where $T_{\text{max}}$ is defined as

$$T_{\text{max}} = \sum_{k=1}^{r+1} k \cdot Q(k).$$

(18)

It can easily be shown that the closed form expression for $T_{\text{max}}$ is

$$T_{\text{max}} = \frac{1 - p^{r+1} \left(1 + (r + 1)(1 - p)\right)}{(1 - p)^2}$$

(19)

where $p = r/N_R$. If the random number is generated using the hardware circuit shown in Fig. 2, then $N_R = 65535$. Hence, it can be assumed that for any practical application of the phase measurement algorithm $2(h_{\text{max}} + n) - 1 << N_R$, i.e., $p << 1$. Thus from (19) it is clear that $T_{\text{max}} \approx 1$, which implies that $\text{TRIAL}_{\text{avg}} \approx 1$.

Without any loss of generality we can say that the same algorithm can be used to measure the phase using the samples of the signals $x(t)$ and $x(t + t_1)$. The only difference is that we must consider $\theta$ as a negative quantity for the signal $x(t + t_1)$. If we want to collect the samples of $x(t)$, the sampling process must be initiated at a positive-zero-cross point (say, the first positive-zero-cross point) and terminated at the second positive-zero-cross point. In order to collect the samples of $x(t + t_1)$, the sampling process must be started after a delay of $t_1$ from the second positive-zero-cross point and terminated after a delay of $t_1$ from the third positive-zero-cross point. The sampling operation for the next trial cannot be started until a delay after the fourth positive-zero-cross point. Thus for every trial with a nonzero $\theta$, a duration of two cycles of the input signal is lost.

It has already been mentioned that for 100-percent reliability $\text{TRIAL}_{\text{max}} = 2(h_{\text{max}} + n)$. If $S_r$ is the minimum sampling rate (samples/cycle of $x(t)$) required to measure the $n$th harmonic phase, then the average time required to measure the phase (phase acquisition time) with 100-percent reliability can be expressed in number of cycles of $x(t)$ as

$$\tau_{\text{avg}} = (S_r - n - 2) \cdot (1 + 2 \cdot \text{TRIAL}_{\text{avg}}) + 4 \cdot (h_{\text{max}} + n) + 1.$$  

(20)

In order to derive (20) it is assumed that for any sampling rate, one cycle is spent to compute the phase with $\theta = 0$ and two cycles are spent for every trial where the phase is computed with a nonzero $\theta$. Thus for a given sampling rate, $1 + 2 \cdot \text{TRIAL}_{\text{avg}}$ cycles are spent to determine that the sampling rate is not high enough to compute the phase. Since the sampling rate is started from $n + 2$ samples/cycle, a duration of $(S_r - n - 2) \cdot (1 + 2 \cdot \text{TRIAL}_{\text{avg}})$ cycles is required for the sampling rates $n + 2$ to $S_r - 1$ samples/cycle. Since for the last sampling rate (i.e., $S_r$) the number of trials will be equal to $\text{TRIAL}_{\text{max}} = 2(h_{\text{max}} + n)$, the number of cycles required for the sampling rate $S_r$ is $4 \cdot (h_{\text{max}} + n) + 1$. Thus (20) is derived. Since $\text{TRIAL}_{\text{avg}} \approx 1$, (20) can be rewritten as

$$\tau_{\text{avg}} \approx 3 \cdot S_r - 3n - 5 + 2 \cdot \text{TRIAL}_{\text{max}}.$$  

(21)

where $\text{TRIAL}_{\text{max}} = 2(h_{\text{max}} + n)$. Hence, when $h_{\text{max}}$ is very large the phase acquisition time will be very high. The measurement process could be made very fast at the cost of momentary reliability. The term momentary reliability is used to mean the reliability at the moment when the phase is accepted as a correct value. The reliability can be increased by continuing the measurement process (after the phase has been accepted as a correct value) with different values of $\theta$, but without changing the sampling rate, and then averaging the measured phases. After the phase is accepted as a correct value for the first time, if the measured phase at any other time is found to be too different from the average phase then the sampling rate has to be increased. Here it is assumed that signal $x(t)$ is stable enough not to have any large change in phase abruptly.

When $\text{TRIAL}_{\text{max}}$ is less than $2(h_{\text{max}} + n)$, the measurement process sometimes may not be reliable due to the fact that during every trial, the value of $\theta$ may be a root of (13). When $\text{TRIAL}_{\text{max}}$ is less than $2(h_{\text{max}} + n)$,
the probability that the value of \( \beta \) is a nonzero root of (13) during all the TRIALmax trials is less than \( p^{\text{TRIALmax}} \), where \( p \) is the same term used in (19). Hence, the probability that the phase is valid when it is accepted as a correct value is greater than \( P_r \), where \( P_r \) is expressed as

\[
P_r = 1 - p^{\text{TRIALmax}}. \tag{22}
\]

Thus the reliability of the measurement process (momentary reliability) is greater than \( P_r \). If \( h_{\text{max}} = 200 \), which is high enough for any practical system, and \( N_k = 65,535 \) then \( p = 0.0061 \) for measuring the fundamental phase \( (n = 1) \). In this case, at least 99.39 percent and 99.9963 percent reliability could be achieved with TRIALmax equal to 1 and 2, respectively. Hence, TRIALmax = 2 could be used for any practical system, and (21) can then be written as

\[
\tau_{\text{avg}} = 3 \cdot S_r - 3n - 1. \tag{23}
\]

The phase acquisition time for measuring the fundamental phase with TRIALmax = 2 could be computed using the value of \( n \) equal to 1 in (23), and this acquisition time could be expressed as

\[
\tau_{\text{avg}} = 3 \cdot S_r - 4. \tag{24}
\]

IV. VERIFICATION OF THE ALGORITHM USING SYNTHESIZED SIGNALS

Synthesized signal \( x(t) \) was generated using the expression shown in (25) to verify the phase measurement algorithm. A simulation software was written using Turbo Pascal 1.1 to run on a Mac II personal computer. The simulation was run only to compute the fundamental:

\[
x(t) = \sum_{n=1}^{10} C_n \sin(n\omega t + \beta_n) \tag{25}
\]

phase \( \beta_n \) of signal \( x(t) \) with the value of TRIALmax equal to 2. The simulation program was run eleven times for different synthesized signals, and the results are shown in Fig. 3. The simulation results show that the fundamental phase could be accurately measured using a sampling rate \( S_s \leq h + 2 \), where \( h \) is the highest harmonic present in the synthesized signal. For example, the result of RUN #4 (see Fig. 3) shows that a sampling rate of 3 samples/cycle could be used to measure the fundamental phase even when the signal \( x(t) \) has 3rd, 6th, and 9th harmonics. This fact can also be verified using (4). From (4) it is seen that if the sampling rate is 3 samples/cycle \( (N = 3) \), and if signal \( x(t) \) has 3rd, 6th, 9th, 12th, etc., harmonics only, then the response of the fundamental component of \( x(i \Delta T) \), \( 0 \leq i \leq N - 1 \) is not affected by any other harmonics present in \( x(t) \).

The value of TRIALavg and phase acquisition time \( \tau \), obtained from the simulation, are found to be approximately the same as those obtained from the analytical expressions.

<table>
<thead>
<tr>
<th>RUN</th>
<th>TRIALavg</th>
<th>( \tau )</th>
<th>( \text{Expression of the Synthesized Signal} )</th>
<th>( \text{Value of TRIALavg} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>1</td>
<td>1.5</td>
<td>( x(t) = \sum_{n=1}^{10} C_n \sin(n\omega t + \beta_n) )</td>
<td>1</td>
</tr>
<tr>
<td>#2</td>
<td>2</td>
<td>1.0</td>
<td>( x(t) = \sum_{n=1}^{10} C_n \sin(n\omega t + \beta_n) )</td>
<td>2</td>
</tr>
<tr>
<td>#3</td>
<td>3</td>
<td>0.5</td>
<td>( x(t) = \sum_{n=1}^{10} C_n \sin(n\omega t + \beta_n) )</td>
<td>3</td>
</tr>
<tr>
<td>#4</td>
<td>4</td>
<td>0.0</td>
<td>( x(t) = \sum_{n=1}^{10} C_n \sin(n\omega t + \beta_n) )</td>
<td>4</td>
</tr>
</tbody>
</table>

V. SIGNIFICANCE OF THE ADAPTIVE SAMPLING ALGORITHM

The novel feature of the adaptive sampling algorithm, developed in this paper, is that it requires a very low sampling rate (at the cost of acquisition time) to measure the phase. The highest harmonic component or the highest frequency component of the signal does not have to be a known quantity. The technique automatically adapts its sampling rate to measure the phase accurately. The major significance of the adaptive sampling is that, although the phase may not be measured by sampling the signal at the maximum sampling rate available for the system, the phase could be measured if the signal is sampled at a lower sampling rate. For example, let the maximum sampling rate available for a system be \( S \) samples/s. Let us also assume that we want to measure the phase of a signal of frequency \( S/4 \) Hz, which is distorted by a third harmonic. From (4) it is clear that if the signal is sampled at the rate 4 samples/cycle (i.e., \( S \) samples/s), the fundamental phase would be affected by the aliasing effect. However, if the signal is sampled at the rate 3 samples/cycle, then there will be no aliasing effect on the fundamental phase, and the phase can be accurately measured.

If a given A/D converter has to be used to build a phase measuring equipment, then a very wide frequency range can be covered using adaptive sampling rather than using conventional sampling technique. The price that we have to pay in adaptive sampling technique is the phase acquisition time.

VI. CONCLUSION

This paper presents a new technique for phase measurement at a low sampling rate. If a phase measuring...
equipment has to be built for a given frequency range then the adaptive sampling technique will require a low cost A/D converter as compared to the A/D converter required for conventional sampling. All the mathematical formulations required to develop this technique have been derived and shown. The analytical expressions for the performance measurement of this technique have also been shown. The technique was verified using synthesized signals. The performance of the technique obtained from the simulation result was found to be approximately the same as those obtained from the analytical expressions. This technique may be used for those applications which do not require very fast phase acquisition time.

**APPENDIX A**

In this appendix we try to find the roots of the following equation:

\[ A_0 + \sum_{n=1}^{M} A_n \cos (n\theta) + \sum_{n=1}^{M} B_n \sin (n\theta) = 0 \]  

(A.1)

Let \( \sin (\theta) = X \), then the following expressions can be easily derived:

\[
\begin{align*}
\sin (n\theta) &= \begin{cases} 
P_n Y, & \text{if } n \text{ is odd} \\ 
Q_{n-1} Y, & \text{if } n \text{ is even}
\end{cases} \tag{A.2a}
\cos (n\theta) &= \begin{cases} 
Q_{n-1} Y, & \text{if } n \text{ is odd} \\ 
P_n, & \text{if } n \text{ is even}
\end{cases} \tag{A.2b}
\end{align*}
\]

where

\[ Y = \sqrt{1 - X^2} \quad \text{and} \quad Q_0 = 1. \]

\( P_n \) and \( Q_n \) are the \( n \)-degree polynomials of \( X \). For example

\[
\begin{align*}
\sin (2\theta) &= 2X(1 - X^2)^{0.5} = P_4 Y, \quad \cos (2\theta) = 1 - 2X^2 = P_2, \\
\sin (3\theta) &= 3X - 4X^3 = P_3, \quad \cos (3\theta) = (1 - 4X^2)(1 - X^2)^{0.5} = Q_4 Y, etc.
\end{align*}
\]

Equation (A.1) can be rewritten as

\[
A_0 + \sum_{n \text{ even}}^{} A_n Q_{n-1} Y + \sum_{n \text{ odd}}^{} A_n P_n + \sum_{n \text{ odd}}^{} B_n P_n + \sum_{n \text{ even}}^{} B_n Q_{n-1} Y = 0. \tag{A.3}
\]

After doing some mathematical manipulations (A.3) can be expressed as

\[
\begin{align*}
&\left( A_0 + \sum_{n \text{ even}}^{} A_n P_n + \sum_{n \text{ odd}}^{} B_n P_n \right)^2 \\
&+ \left( \sum_{n \text{ odd}}^{} A_n Q_{n-1} + \sum_{n \text{ even}}^{} B_n Q_{n-1} \right)^2 Y^2 = 0
\end{align*}
\]

or

\[ R_{2M} + S_{2M} = 0 \]  

(A.4)

where

\[
\begin{align*}
R_{2M} &= \left( A_0 + \sum_{n \text{ even}}^{} A_n P_n + \sum_{n \text{ odd}}^{} B_n P_n \right)^2 \\
S_{2M} &= \left( \sum_{n \text{ odd}}^{} A_n Q_{n-1} + \sum_{n \text{ even}}^{} B_n Q_{n-1} \right)^2 Y^2.
\end{align*}
\]

\( R_{2M} \) and \( S_{2M} \) are 2M-degree polynomials of \( X \), which means that (A.4) has 2M roots in \( X \). Thus the roots of (A.1) can be written as \( \sin (\theta) = x_i, \quad i = 1, 2, 3, \ldots, 2M \). Although there are two values of \( \theta \) (\( \theta \) and \( \pi - \theta \)), which satisfy \( \sin (\theta) = x_i \) for any value of \( i \), only one value of \( \theta \) is the root of (A.1), because \( \cos (\theta) \neq \cos (\pi - \theta) \). Thus (A.1) has 2M roots in terms of \( \theta \). Some of these roots may be complex numbers. Hence, the maximum number of real roots of (A.1) in terms of \( \theta \) is 2M. Since we have assumed that \( h_{\max} \) is the highest harmonic that can ever occur in signal \( r(t) \), represented by (1), the value of \( i \) in (13) can be in the range \(-h_{\max} \leq i \leq h_{\max} \). Hence, the maximum value of \( |i - n| \) in (13) could be \( h_{\max} + n \). Thus the maximum number of real roots of (13) in terms of \( \theta \) is \( 2(h_{\max} + n) \). Since \( \theta = 0 \) is a trivial solution of (13), the maximum number of nonzero roots of (13) is \( 2(h_{\max} + n) - 1 \). This is what we wanted to prove.

**REFERENCES**


